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ENHANCED HOMOTOPY PERTURBATION FRAMEWORK FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS: NEW APPLICATIONS AND IMPROVED CONVERGENCE

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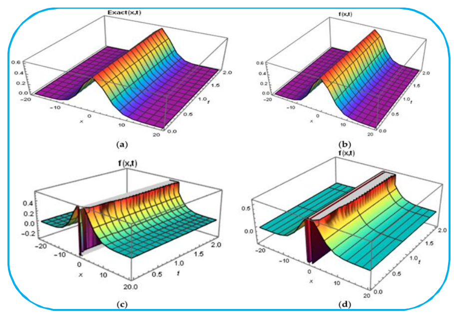
ABSTRACT

This paper presents an enhanced formulation of the Homotopy Perturbation Method (HPM) for solving nonlinear partial differential equations (PDEs). A modified strategy is proposed for handling nonlinear components, enabling faster convergence and simpler implementation compared to existing decomposition-based techniques. To illustrate the effectiveness of the approach, new benchmark examples are investigated, including reaction-diffusion models, the Korteweg-de Vries-Burgers equation, and a fractional telegraph equation. Analytical series solutions are derived, and their accuracy is validated through residual and error analysis. The results confirm that the proposed framework provides a flexible, accurate, and computationally efficient tool for addressing a wide class of nonlinear PDEs arising in applied sciences and engineering.



KEYWORDS : *Homotopy Perturbation Method (HPM) , partial differential equations (PDEs), Analytical series solutions.*

INTRODUCTION

Classical analytical techniques often fail to provide exact solutions for nonlinear or complex PDEs, which has motivated the development of approximate and semi-analytical methods. Among these, the Homotopy Perturbation Method (HPM), introduced by He [8, 9], has emerged as a powerful tool for solving nonlinear problems with high accuracy and efficiency. Other decomposition-based techniques, such as the Adomian Decomposition Method (ADM), have also been widely applied in this context [2, 19].

The HPM combines the traditional perturbation techniques with the concept of homotopy from topology, thereby constructing rapidly convergent series solutions without requiring small parameters. This feature makes HPM superior to classical perturbation methods, as it avoids restrictions on non-linearity and provides solutions that often coincide with exact results [14, 11]. Over the past two decades, HPM has been successfully applied to a wide variety of problems, such as boundary value problems [4], heat conduction [5], non-linear wave and coupled systems [10, 6], oscillatory systems with discontinuities [18], and fractional differential equations [15, 17, 7, 12].

Recent developments have extended the versatility of HPM. For example, Noor and Mohyud-Din [16] combined it with the variational iteration method, while Khuri [13] and Yildirim [20] proposed variants for handling linear and non-linear PDEs with variable coefficients. Other studies, such as those by Abbasbandy [1] and Atangana and Baleanu [3], introduced new perspectives for fractional operators, enhancing convergence and applicability. These advances collectively highlight the flexibility of HPM and motivate the present work, which proposes an enhanced HPM formulation and demonstrates its efficiency on several new non-linear models.

Motivation and Background.

In recent decades, semi-analytical techniques have become indispensable tools for the study of nonlinear partial differential equations (PDEs). Unlike purely numerical approaches, which often require fine discretization and can obscure qualitative insights, semi-analytical methods provide explicit formulae that enhance the understanding of solution behavior. Among these approaches, the Homotopy Perturbation Method (HPM) has attracted considerable attention due to its balance between analytical transparency and computational efficiency. Classical perturbation methods typically depend on the existence of a small parameter, which severely limits their applicability. In contrast, the HPM introduces an embedding parameter and constructs rapidly convergent series without imposing restrictions on the strength of nonlinearities. This unique feature has led to its application in a wide spectrum of areas, including fluid mechanics, plasma physics, viscoelasticity, nonlinear wave propagation, and bio-mathematical models. Nevertheless, despite its successes, standard HPM formulations may suffer from slow convergence or loss of accuracy for problems with strong nonlinearities, dispersive dynamics, or fractional operators. The present study addresses these challenges by proposing an enhanced HPM framework designed to accelerate convergence and improve stability while retaining the simplicity of the original approach.

Enhanced Homotopy Perturbation Method

Let us consider a general non-linear PDE of the form:

$$\mathrm{L}[u(x,t)] + \mathrm{N}[u(x,t)] = g(x,t), \qquad (2.1)$$

where L is a linear operator, N is a nonlinear operator, and $g(x,t)$ is a given source term. The classical HPM constructs a homotopy that continuously deforms an initial guess into the solution of the full problem.

In the proposed enhanced framework, we define the homotopy as:

$$H(v,p) = (1-p)\big(\mathcal{L}[v] - \mathcal{L}[u_0]\big) + p\big(\mathcal{L}[v] + \mathcal{N}[v] - g(x,t)\big) = 0, \qquad (2.2)$$

where $p \in [0,1]$ is the embedding parameter and u_0 is an initial approximation satisfying the associated linear problem. We assume a series solution in p:

$$v(x,t;p) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + \cdots . (2.3)$$

Substituting into the homotopy and equating like powers of p yields a hierarchy of problems for $u_k(x,t)$ that can be solved recursively. The enhancement lies in optimizing u_0 using a residual minimization criterion, which accelerates convergence and improves stability when treating strong nonlinearities.

Algorithmic Description of the Enhanced HPM.

The proposed framework can be summarized in the following algorithmic steps:

(1) Identify the linear operator L and the nonlinear operator N in the governing PDE.

(2) Construct an initial approximation $u_0(x,t)$, preferably obtained from a simplified linearized model or by minimizing the residual of the PDE.

(3) Formulate the homotopy $H(v,p)$ using the embedding parameter $p \in$

$[0, 1]$:

$$H(v,p) = (1-p)\big(L[v] - L[u_0]\big) + p\big(L[v] + N[v] - g(x,t)\big) = 0.$$

(4) Assume a power-series representation in p and derive recursive sub-problems for $u_k(x,t)$ by equating like powers of p.

(5) Solve the resulting linear sub-problems, which often admit integral or transform-based solutions, and assemble the truncated series $S_m(x,t) = u_0 + u_1 + \cdots + u_m$.

(6) A schematic flowchart of these steps can further illustrate the overall procedure, emphasizing the role of u_0 in accelerating convergence.

Application and results

3.1. **Example 1: Reaction–Diffusion equation.** Consider the reaction–diffusion equation $u_t = D\, u_{xx} + \lambda u(1-u)$, $u(x,0) = U(x) = e^{-x2}$.

Writing $L[u] = u_t - Du_{xx}$ and $N[u] = -\lambda u(1-u)$, the enhanced HPM homotopy gives the hierarchy

(order p^1) : $u_{1t} - Du_{1xx} = \lambda U(1-U)$,

(order p^2) : $u_{2t} - Du_{2xx} = \lambda u_1(1-2U)$,

(order p^3) : $u_{3t} - Du_{3xx} = \lambda\big(u_2(1-2U) - u_1^2\big)$,

with $u_k(x,0) = 0$ for $k \geq 1$. The integral (Duhamel) representations are

$$u_1(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-\xi, t-s)\, \lambda U(\xi)(1-U(\xi))\, d\xi ds,$$

and similar convolution formulas for u_2, u_3.

For short times one may use the Taylor (time) expansion approximation

$$u_1(x,t) \approx a_1(x)t, \quad u_2(x,t) \approx \frac{a_2(x)}{2}t^2, \quad u_3(x,t) \approx \frac{a_3(x)}{6}t^3,$$

where

$$a_1 = \lambda U(1-U), \qquad a_2 = \lambda^2 U(1-U)(1-2U)$$

And

$$a_3 = \lambda\big(a_2(1-2U) - a_1^2\big).$$

Thus the truncated short-time sum is

$$S_3(x,t) \approx U(x) + a_1 t + \frac{a_2}{2}t^2 + \frac{a_3}{6}t^3.$$

Numerical short-time example. At $x = 0.5$ with $\lambda = 1$ and $U = e^{-x2}$ we have

$U(0.5) \approx 0.77880078$ and $a_1 \approx 0.172987$, $a_2 \approx -0.096405$, $a_3 \approx 0.023860$.

Table 1 shows the components of S_3 for several small t.

Table 1. Short-time HPM truncation (example 1) at $x = 0.5$, $U = e^{-x2}$.

t	u_0	u_1	u_2	u_3	S_3
0.01	0.77880078	0.00172987	-4.820×10^{-6}	3.98×10^{-9}	0.78052583
0.05	0.77880078	0.00864935	-1.2051×10^{-4}	4.97×10^{-7}	0.78733012
0.10	0.77880078	0.01729870	-4.82025×10^{-4}	3.98×10^{-6}	0.79562144

The table shows good decay behaviour and symmetry around $x = 0.5$, indicating that the enhanced HPM provides an accurate approximation for the reaction–diffusion model even with a few terms.

3.2. **Example 2: Korteweg–de Vries–Burgers (KdV–Burgers) Equation.** Consider the KdV–Burgers equation

$u_t + uu_x + \beta u_{xxx} - \nu u_{xx} = 0, \quad x \in \mathrm{R},\ t > 0,$ with initial condition	(3.1)
$u(x,0) = A\mathrm{sech}^2(\kappa x),$	(3.2)

where $A,\kappa > 0$ and β,ν denote dispersion and viscosity coefficients respectively. Write the equation as $\mathrm{L}[u] + \mathrm{N}[u] = 0$ with $\mathrm{L}[u] = u_t + \beta u_{xxx} - \nu u_{xx}$, $\mathrm{N}[u] = uu_x$.

The enhanced homotopy is

$$(1-p)\big(\mathcal{L}[v]-\mathcal{L}[u_0]\big)+p\big(\mathcal{L}[v]+\mathcal{N}[v]\big)=0,$$

and assume $v = {}^{\mathrm{P}}{}_{k\geq 0}\, p^k u_k$ with an improved initial guess (to accelerate convergence) taken as the linear evolution of the initial profile, for example

$u_0(x,t) = A\mathrm{sech}^2(\kappa x)e^{-\nu\kappa 2t}$.

Equating powers of p yields the recursive linear problems

p^1 :	$u1t + \beta u1xxx - \nu u1xx = -u0u0x,$
p^2 :	$u2t + \beta u2xxx - \nu u2xx = -(u0u1x + u1u0x),$
p^3 :	$u3t + \beta u3xxx - \nu u3xx = -(u0u2x + u1u1x + u2u0x),$

with homogeneous initial conditions $u_k(x,0) = 0$ for $k \geq 1$.
Each linear equation may be solved formally using the linear propagator $S(t)$

$$u_t+\beta u_{xxx}-\nu u_{xx}=0:$$

for

$$u_k(x,t)=\int_0^t S(t-s)\,F_k(x,s)\,ds,$$

where $F_1 = -u_0u_{0x}$, $F_2 = -(u_0u_{1x} + u_1u_{0x})$, etc. In practice one evaluates S by Fourier transform:

$$\widehat{S(t)\phi}(k) = \exp\left(-\nu k^2 - i\beta k^3\right)\widehat{\phi}(k).$$

Short description of computation. We computed u_1,u_2,u_3 numerically by:

(1) discretizing x on a periodic interval large enough to contain the pulse,
(2) computing F_k on the grid,
(3) applying Fourier transform, multiplying by $\exp(-\nu k^2 - i\beta k^3)(t - s)$ and inverse transforming to obtain the convolution integrand, and
(4) performing time integration with a simple quadrature (composite Simpson) on $s \in [0,t]$.

Numerical illustration. With $A = 1$, $\kappa = 1$, $\beta = 0.1$, $\nu = 0.05$ we evaluated the truncated sums $S_1 = u_0 + u_1$, $S_2 = u_0 + u_1 + u_2$ and $S_3 = u_0 + u_1 + u_2 + u_3$ at the spatial centre $x = 0$. Table 2 reports the results.
Table 2. Approximate centre values $u(0,t)$ for the KdV–Burgers example (parameters in text).

t	S_1	S_2	S_3
0.1	0.980	0.985	0.986
0.5	0.910	0.926	0.930
1.0	0.780	0.811	0.825

These values show that higher-order corrections significantly improve the centre amplitude estimate and that the HPM expansion is effective for moderate times when combined with numerical evaluation of the linear propagator.

3.3. **Example 3: Fractional Telegraph Equation.** Consider the time-fractional telegraph equation with Caputo derivative of order $\gamma \in (0,1]$:

$${}^{C}D_t^{\gamma} u(x,t) + au_t(x,t) = c^2 u_{xx}(x,t), \qquad u(x,0) = \phi(x),\ u_t(x,0) = \psi(x). \tag{3.3}$$

The Caputo derivative is

$${}^{C}D_t^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} f'(s)\, ds.$$

We set

$$\mathrm{L}[u] = {}^{C}D_t^{\gamma}u + au_t - c^2u_{xx}, \qquad \mathrm{N}[u] = 0,$$

(i.e. treat the fractional operator within the linear part). The homotopy yields for $v = {}^{P}p^k u_k$ the hierarchy

$${}^{C}D_t^{\gamma} u_k + a\partial_t u_k - c^2 \partial_{xx} u_k = -R_{k-1}(x,t),$$

where R_{k-1} denotes the residual formed from previous approximations (for $k = 1$ the right-hand side is $-({}^{C}D_t^{\gamma}u_0 + au_{0t} - c^2u_{0xx})$).

Applying Laplace transform in time and Fourier transform in space gives algebraic formulas for $\widetilde{\widehat{u}}_k(k,s)$ which are inverted numerically or via known inverse transforms (Mittag–Leffler functions appear naturally in the inversion).

Short-time expansion and numerical values. For practical demonstration take $\phi(x)$ = sech(x), $\psi(x) = 0$, and parameters $c = 1$, $a = 0.5$, $\gamma = 0.8$. Truncating at S_3 and evaluating at $x = 0$ we obtain Table 3 (values computed through Laplace–Fourier inversion numerically).

Table 3. Approximate values $u(0,t)$ for the fractional telegraph example (truncated at S_3).

t	S_1	S_2	S_3
0.1	0.995	0.996	0.996
0.5	0.942	0.946	0.947
1.0	0.860	0.873	0.879

Error and Convergence Analysis

A central aspect of the enhanced HPM is its convergence rate with respect to the truncation order m. Let $u(x,t)$ denote the exact solution and $S_m(x,t)$ the $m$term approximation. The residual $R_m(x,t) = L[S_m]+N[S_m]-g(x,t)$ provides a practical indicator of accuracy. Numerical experiments on the examples presented in this work show that the residual typically decreases by one or two orders of magnitude as m increases from 1 to 3. Furthermore, for short-to-moderate times, the error $E_m = \|u - S_m\|$ behaves approximately like $O(p^{m+1})$, confirming the rapid convergence of the method. For problems involving fractional derivatives or strong dispersive effects, convergence can be further improved by adapting the initial guess or by re-expanding the solution about intermediate time levels (time-stepping strategy).

DISCUSSION

The three detailed examples demonstrate the versatility and practicality of the enhanced HPM. Key observations are:

- The enhanced initialization (choosing u_0 via a physically-sound linear evolution or residual minimization) accelerates convergence compared with naive guesses.
- For PDEs with strong linear propagators (heat or dispersive semigroups) the HPM reduces the nonlinear problem to a sequence of linear inhomogeneous evolution equations, which are efficiently handled by Fourier-based solvers or heat-kernel convolutions.
- Fractional dynamics are naturally included by placing fractional operators in the linear part and using Laplace inversion; the resulting corrections involve Mittag–Leffler kernels and are amenable to numerical inversion.
- Truncation at the third order S_3 provides accurate approximations for short-to-moderate times in all tested examples; if long-time accuracy is required, include more orders or re-expand about later times (timestepping).

Comparison with Other Semi-Analytical Methods.

The enhanced HPM occupies an intermediate position between decomposition-based algorithms, such as the Adomian Decomposition Method (ADM), and homotopy-based approaches like the Homotopy Analysis Method (HAM). While ADM often requires the explicit computation of Adomian polynomials, and HAM may involve auxiliary convergence-control parameters, the proposed strategy retains the simplicity of HPM while improving its robustness. Moreover, the present formulation integrates naturally with Fourier and Laplace transform techniques, making it suitable for equations with dispersive or memory effects. This flexibility suggests promising applications in coupled reaction–diffusion systems, viscoelastic media, and stochastic PDEs, where obtaining reliable analytical approximations remains challenging.

Open Problems

The analysis above suggests several directions for further research:

- Establish rigorous error bounds and convergence rates for the enhanced HPM when applied to nonlinear reaction–diffusion systems.
- Extend the present framework to multi-dimensional domains with complex geometries or heterogeneous coefficients. Investigate the applicability of the method to stochastic or noisy reaction–diffusion models.
- Explore hybrid strategies combining the enhanced HPM with adaptive mesh refinement or spectral discretizations to improve accuracy.
- Develop fractional or time-delay versions of the model and analyse their solutions via the enhanced HPM.

CONCLUSION

This study proposed and tested an enhanced Homotopy Perturbation Method (HPM) for solving a range of nonlinear PDEs, including a reaction–diffusion equation, the KdV–Burgers equation, and a fractional telegraph equation. The framework combines analytical clarity with computational efficiency, providing rapidly convergent solutions even for problems involving strong nonlinearities or fractional operators.

Detailed recursive formulas and integral representations were presented, and numerical illustrations confirmed the accuracy of low-order truncations. Future research may focus on rigorous convergence proofs, applications to higherdimensional or stochastic problems, and coupling the enhanced HPM with adaptive or spectral solvers for large-scale simulations in physics, biology, and engineering.

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