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DIGITAL CONTRACTIVE TYPE MAPPINGS IN DIGITAL METRIC SPACES

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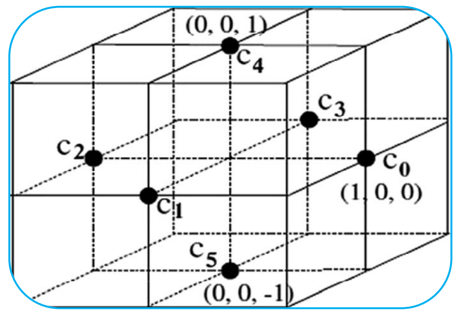
ABSTRACT:
In the framework of digital metric spaces, the study present digital β and ψ contractive mappings in this paper. Finally, it goes over a few examples to show our findings. A fascinating topic for dynamic study in non-linear analysis is fixed point theory. The fact that a unit closed ball in R^2 has a fixed point was demonstrated by Brouwer in 1912. In 1922, Banach presented the most noteworthy outcome in the fixed-point theory. He demonstrated that there is a distinct fixed point for every contraction in a complete metric space. The Banach fixed point theorem was further developed in a variety of ways by other writers. An author recently presented the concept of σ-ψ contractive mappings and demonstrated the associated fixed-point theorems.



KEYWORDS: *digital metric spaces, contractive mapping, demonstrated the associated fixed-point theorems.*

INTRODUCTION

Fixed point theory is a beautiful subject for dynamic research in non-linear analysis. In 1912, Brouwer proved a result that a unit closed ball in $\mathbb{R}^n$ has a fixed point. The most remarkable result in the fixed point theory was given by Banach in 1922. He proved that each contraction in a complete metric space has a unique fixed point. Later on, many authors generalized the Banach fixed point theorem in various ways. Recently, Samet et al. introduced the notion of $\alpha - \psi$ contractive mappings and proved the related fixed point theorems.

Digital topology is a developing area based on general topology and functional analysis which studies features of 2D and 3D digital images. Rosenfield was the first to consider digital topology as the tool to study digital images. Kong, then introduced the digital fundamental group of a discrete object. The digital versions of the topological concepts were given by Boxer, who later studied digital continuous functions. Later, he gave results of digital homology groups of 2D digital images in Ege and Karaca give relative and reduced Lefschetz fixed point theorem for digital images. They also calculate degree of antipodal map for the sphere like digital images using fixed point properties. Ege and Karaca then defined a digital metric space and proved the famous Banach Contraction Principle for digital images.

In this paper we generalize the concept of $\alpha - \psi$-contractive mappings in the setting of digital metric space, as $d - \beta - \psi$-contractive mappings.

Preliminaries

Definition 1.1: Let Ψ be a family of functions $\psi: [0,\infty) \to [0,\infty)$ satisfying the following conditions:
(i) ψ is nondecreasing;
(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that
$$\psi^{k+1}(t) \le a\psi^k(t) + v_k$$
For $k \ge k_0$ and any $t \in \mathbb{R}^+$.

Lemma 1.2: If $\psi \in \Psi$, then the following hold:
(i) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$;
(ii) $\psi(t) < t$ for any $t \in (0,\infty)$;
(iii) ψ is continuous at 0 ;
(iv) the series $\sum_{n=1}^{\infty} \psi^k(t)$ converges for any $t \in \mathbb{R}^+$.

Recently, Samet et al. introduced the following concepts.

Definition 1.3: Let $T: X \to X$ and $\alpha: X \times X \to [0,\infty)$. we say that T is α admissible if for all $x, y \in X$, we have
$$\alpha(x,y) \ge 1 \Rightarrow \alpha(Tx,Ty) \ge 1$$
Definition 1.4: Let (X, d) be a metric space and let $T: X \to X$ be a given mapping. We say that T is an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha: X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that
$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y))$$
for all $x, y \in X$.

Clearly, any contractive mapping, that is, a mapping satisfying Banach contraction, is an $\alpha - \psi$-contractive mapping with $\alpha(x,y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, for all $t > 0$ and some $k \in [0,1)$.

Let X be a subset of $\mathbb{Z}^n$ for a positive integer n where $\mathbb{Z}^n$ is the set of lattice points in the n-dimensional Euclidean space and ρ represent an adjacency relation for the members of X. A digital image consists of (X, ρ).

Definition 1.5: Let l, n be positive integers, $1 \le l \le n$ and two distinct points
$$a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in \mathbb{Z}^n$$
a and b are k_l - adjacent if there are at most l indices i such that $|a_i - b_i| = 1$ and for all other indices j such that $|a_j - b_j| \ne 1, a_j = b_j$.
There are some statements which can be obtained from definition 2.1:

- a and b are 2- adjacent if $|a - b| = 1$.
- a and b in $\mathbb{Z}^2$ are 8 - adjacent if they are distinct and differ by at most 1 in each coordinate.
- a and b in $\mathbb{Z}^3$ are 26- adjacent if they are distinct and differ at most 1 in each coordinate.
- a and b in $\mathbb{Z}^3$ are 18 - adjacent if are 26 - adjacent and differ by at most two coordinates.
- a and b are 6-adjacent if they are 18- adjacent and differ in exactly one coordinate.

A ρ-neighbour [9] of $a \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is ρ-adjacent to a where $\rho \in \{2,4,8,6,18,26\}$ and $n \in 1,2,3$. The set
$$\mathrm{N}_\rho(a) = \{b \mid b \text{ is } \rho - \text{ adjacent to } a\}$$
is called the ρ-neighbourhood of a. A digital interval [9] is defined by
$$[p,q]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid p \le z \le q\}$$
where, $p, q \in \mathbb{Z}$ and $p < q$.

A digital image $X \subset \mathbb{Z}^n$ is ρ - connected [10] if and only if for every pair of different points $u, v \in X$, there is a set $\{u_0, u_1, \dots, u_r\}$ of points of digital image X such that $u = u_0, v = u_r$ and u_i and u_{i+1} are ρ - neighbours where $i = 0,1, \dots, r-1$.

Definition 1.6: Let $(X, \rho_0) \subset \mathbb{Z}^{n_0}, (Y, \rho_1) \subset \mathbb{Z}^{n_1}$ be digital images and $T: X \to Y$ be a function.

- T is said to be (ρ_0, ρ_1) – continuous, if for all ρ_0 - connected subset E of $X, f(E)$ is a ρ_1 - connected subset of Y.

- For all ρ_0 - adjacent points $\{u_0, u_1\}$ of X, either $T(u_0) = T(u_1)$ or $T(u_0)$ and $T(u_1)$ are a ρ_1-adjacent in Y if and only if T is (ρ_0, ρ_1)-continuous [9].
- If f is (ρ_0, ρ_1)- continuous, bijective and T^{-1} is (ρ_1, ρ_0)- continuous, then T is called (ρ_0, ρ_1)-isomorphism [11]and denoted by $X \cong_{(\rho_0,\rho_1)} Y$.

A $(2, \rho)$- continuous function T, is called a digital ρ - path [9] from u to v in a digital image X if $T: [0, m]_{\mathbb{Z}} \to X$ such that $T(0) = u$ and $T(m) = v$. A simple closed ρ - curve of $m \geq 4$ points [12] in a digital image X is a sequence $\{T(0), T(1), \ldots, T(m-1)\}$ of images of the ρ - path $T: [0, m-1]_{\mathbb{Z}} \to X$ such that $T(i)$ and $T(j)$ are ρ - adjacent if and only if $j = i \pm \text{mod}m$.

Definition 1.7: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) is a Cauchy sequence if for all $\in > 0$, there exists $\delta \in \mathbb{N}$ such that for all $n, m > \delta$, then

$$d(x_n, x_m) < \epsilon$$

Definition 1.8: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $p \in X$ if for all $\in > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n > \delta$, then

$$d(x_n, p) < \epsilon$$

Definition 1.9: A digital metric space (X, d, ρ) is a digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, d, ρ) converges to a point p of (X, d, ρ).

Definition 1.10: Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\alpha \in (0,1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y),$$

then T is called a digital contraction map.

Proposition 1.11: Every digital contraction map is digitally continuous.

Theorem 1.12: (Banach Contraction principle) Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $T: X \to X$ be a digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $p \in X$ such that $f(p) = p$.

Main Results

We introduce the concept of digital- $\beta - \psi$-contractive mapping as follows:

Definition 1.1: Let (X, d, ρ) be a digital metric space and let $T: X \to X$ be a given mapping. We say that T is a digital- $\beta - \psi$-contractive mapping if there exist two functions $\beta: X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\beta(x, y) d(Tx, Ty) \leq \psi\big(d(x, y)\big) \tag{1}$$

Definition 1.2: Let $T: X \to X$ and $\beta: X \times X \to [0, \infty)$. We say that T is a β admissible if for all $x, y \in X$, we have

$$\beta(x, y) \geq 1 \Rightarrow \beta(Tx, Ty) \geq 1$$

Theorem 1.3: Let (X, d, ρ) be a complete digital metric space and let $T: X \to X$ is a digital- $\beta - \psi$-contractive mapping and satisfies the following conditions:

(i) T is a β-admissible;
(ii) there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$;
(iii) T is digital continuous.

Then there exists $u \in X$ such that $Tu = u$.

Proof: Let $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$ (such a point exist from the condition (ii)). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_{n0} = x_{no+1}$ for some n_0, then $u = x_{n0}$ is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$ for all n. Since T is β-admissible, we have

$$\beta(x_0, x_1) = \beta(x_0, Tx_0) \geq 1 \Rightarrow \beta(Tx_0, Tx_1) = \beta(x_1, x_2) \geq 1.$$

Inductively, we have

$$\beta(x_n, x_{n+1}) \geq 1 \text{ for all } n = 0,1,2 \ldots\ldots. \tag{2}$$

From (2) and (6), it follows that for all $n \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \beta(x_{n-1}, Tx_{n-1}) d(Tx_{n-1}, Tx_n) \\ &\leq \psi\big(d(x_{n-1}, x_n)\big) \end{aligned} \tag{4}$$

$$d(x_n, x_{n+1}) \leq \psi^n\big(d(x_0, x_1)\big) \text{ for all } n \geq 1$$

Using (5) , we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\quad + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k\big(d(x_0, x_1)\big) \end{aligned}$$

Since $\psi \in \Psi$ and $d(x_0, x_1) > 0$, by lemma 1.2, we get $\sum_{k=1}^{\infty} \psi^k d(x_0, x_1) < \infty$

Thus, we have $\lim_{n,m\to 0} d(x_n, x_m) = 0$.

This implies that $\{x_n\}$ is a digital Cauchy sequence in the complete digital metric space (X, d, ρ). Since (X, d, ρ) is complete, there exist $u \in X$ such that $\{x_n\}$ is digital convergent to u. Since T is digital continuous, it follows that $\{Tx_n\}$ is digital convergent to Tu. By the uniqueness of the limit, we get $u = Tu$, that is, u is a fixed point of T.

The next theorem does not require continuity.

Theorem 1.5: Let (X, d, ρ) be a complete digital metric space. Suppose that $T: X \to X$ is a digital- $\beta - \psi$-contractive mapping and the following conditions satisfies:

(i) T is β-admissible;

(ii) there exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$;

(iii) If $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\}$ is a digital convergent to $x \in X$, then $\beta(x_n, x) \geq 1$ for all n.

Then there exist $u \in U$ such that $Tu = u$.

Proof: Following the proof of theorem 3.4, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is a Cauchy sequence in the complete metric space (X, d, ρ) that is digital convergent to $u \in X$, From (6) and (iii), we have

$$\beta(x_n, u) \geq 1 \text{ for all } n \geq 0. \tag{5}$$

Using the basic properties of digital metric together with (2) and (5), we have

$$\begin{aligned} d(x_{n+1}, Tu) &= d(Tx_n, Tu) \\ &\leq \beta(x_n, u) d(Tx_n, Tu) \\ &\leq \psi\big(d(x_n, u)\big) \end{aligned}$$

Letting $n \to \infty$, and since ψ is continuous at $t = 0$, it follows that

$$d(u, Tu) = 0$$

By definition, we obtain $u = Tu$

With the help of the following example, we show that the hypotheses in theorems 1.4 and 1.5 do not guarantee uniqueness of the fixed point.

Example 1.6: Let $X = [0, \infty)$ be the digital metric space, where $d(x, y) = |x - y|$ for all $x, y \in X$. Consider the self- mapping $T: X \to X$ given by

$$Tx = \begin{cases} 2x - \frac{7}{4} \text{ if } x > 1, \\ \frac{x}{4} \text{ if } 0 \leq x \leq 1. \end{cases}$$

Notice that 1.16, a characterization of the Banach fixed point theorem, cannot be applied in this case because $d(T1, T2) = 2 > 1 = \mathrm{d}(1,2)$.

Define β: $X \times X \to [0, \infty)$, as

$$\beta(x,y) = \begin{cases} 1 \text{ if } x, y \in [0,1] \\ 0 \text{ otherwise} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$. Then, we conclude that T is a digital- $\beta - \psi$-contractive mapping. In fact, for all $x, y \in X$, we have

$$\beta(x,y)d(Tx,Ty) \leq \frac{1}{2}d(x,y).$$

On the other hand, there exists $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$. Indeed, for $x_0 = 1$, we have $\beta(1, T1) = \beta\left(1, \frac{1}{4}\right) = 1$

Notice also that T is continuous. To show that T satisfies all the hypotheses of Theorem 1.4, it is sufficient to observe that T is β-admissible. For this purpose, let $x, y \in X$ such that $\beta(x,y) \geq 1$, which is equivalent to saying that $x, y \in [0,1]$. Due to the definitions of β and T, we have

$$Tx = \frac{x}{4} \in [0,1], Ty = \frac{y}{4} \in [0,1].$$

Hence, $\beta(Tx, Ty) \geq 1$. As a result, all the conditions of theorem 1.4 are satisfied. Note that theorem 1.4 guarantees the existence of a fixed point but not the uniqueness. In this example 0 and $\frac{7}{4}$ are two fixed points of T.

In the following example T is not continuous.

Example 1.7: Let X, d and β be defined as in example 3.6. Let $T: X \to X$

$$Tx = \begin{cases} 2x - \frac{7}{4} \text{ if } x > 1 \\ \frac{x}{3} \text{ if } 0 \leq x \leq 1 \end{cases}$$

let $\psi(t) = \frac{t}{3}$ for $t \geq 0$. Then we conclude that T is a digital- $\beta - \psi$-contractive mapping. In fact, for all $x, y \in X$, we have

$$\beta(x,y)d(Tx,Ty) \leq \frac{1}{2}d(x,y)$$

Furthermore, there exist $x_0 \in X$ such that $\beta(x_0, Tx_0) \geq 1$. For $x_0 = 1$, we have $\beta(1, T1) = \beta\left(1, \frac{1}{3}\right) = 1$.

Let $\{x_n\}$ be a sequence such that $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and as $x_n \to x$ as $n \to \infty$. by the definition of β, we have $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. then we see that $x_n \in [0,1]$. Thus, $\beta(x_n, x) \geq 1$.

To show that T satisfies all the hypotheses of Theorem 2.5, it is sufficient to observe that T is β-admissible. For this purpose, let $x, y \in X$ such that $\beta(x,y) \geq 1$. It is equivalent to saying that $x, y \in [0,1]$. Due to the definition of β and T, we have

$$Tx = \frac{x}{3} \in [0,1], Ty = \frac{y}{3}[0,1].$$

Hence $\beta(Tx, Ty) \geq 1$.

As a result, all the conditions of the theorem 1.5 are satisfied. In this example, 0 and $\frac{7}{4}$ are two fixed points of T.

Theorem 1.8: Adding the following condition to the hypotheses of theorem 1.4 and 1.5 we obtain the uniqueness of a fixed point of T.

For all $x, y \in X$, there exist $z \in X$ such that $\beta(x,z) \geq 1$ and $\beta(y,z) \geq 1$.

Proof: Let $u, u^* \in \mathrm{Fix}(T)$ then by the given condition $\beta(u,z) \geq 1$ and $\beta(u^*, z) \geq 1$

Since T is β-admissible, we get by induction that

$$\beta(u, T^n z) \geq 1\beta(u^*, T^n z) \geq 1 \text{ for all } n = 1,2,3, \dots \#(6)$$

From (9) and (5), we have

$$\begin{aligned} d(u, T^n z) &= d\big(Tu, T(T^{n-1}z)\big) \\ &\leq \beta(u, T^{n-1}z) d\big(Tu, T(T^{n-1}z)\big) \\ &\leq \psi\big(d(u, T^{n-1}z)\big) \end{aligned}$$

Thus, we get by induction that

$$d(u, T^n z) \leq \psi^n(d(u, z)) \text{ for all } n = 1,2,3, \dots$$

Letting $n \to \infty$, and since $\psi \in \Psi$, we have

$$d(u, T^n z) \to 0.$$

This implies that $\{T^n z\}$ is digital convergent to u. Similarly, we get $\{T^n z\}$ is digital convergent to u^*. By the uniqueness of the limit, we get $u = u^*$, that is, the fixed point of T is unique.

Conclusion:

Since digital contractive mappings apply the well-known Banach contraction principle to the non-Euclidean, integer-based character of digital spaces, they are essential for researching discrete systems. They define contractions using digital measurements like the shortest route distance rather than the conventional Euclidean distance. The important realization is that a mapping that moves points "closer" together in this particular metric would ultimately converge to a single fixed point, even in a discrete space. There are important applications for this framework. It is employed in fields such as digital image processing, where algorithms frequently have to identify a particular pixel or area that doesn't change despite undergoing a number of changes. It may also be used to analyse the stability of routing algorithms or iterative processes in computer graphics and network theory. The theory offers a solid mathematical foundation for demonstrating that some discrete processes will always come to an end and result in a reliable, consistent outcome. The main finding is that digital contractive type mappings effectively connect discrete computational issues with continuous fixed-point theory. They provide a strong and sophisticated way to guarantee stability and convergence in a variety of computational and digital applications.

References

1. Brouwer LES, "Uber Abbildungen Von Mannigfaltigkeiten", Math. Ann., 77 (1912), 97-115.
2. Boxer L, Digital Products, Wedges and Covering Spaces, J. Math. Imaging Vis., 25(2006), 159171.
3. Ege O, Karaca I, Banach Fixed Point Theorem for Digital Images, J. Nonlinear Sci. Appl., 8(2015), 237-245.
4. Herman GT, Oriented Surfaces in Digital Spaces, CVGIP: Graphical Models and Image Processing, 55(1993), 381-396.
5. Kannan R., "Some results on fixed points", Bull. Cal. Math. Soc., 60 (1968), 71-76.
6. Osilike M.O., "Stability Results For Fixed Point Iteration Procedures", J. The Nigerian Math. Society, 14 (15) (1995), 17-29.
7. Rhoades B.E., "A fixed point theorem for generalized metric spaces", Internat. J. Math. and Math. Sci., 19 (3) (1996), 457-460.
8. Zamfirescu T., "Fixed Point Theorems In Metric Spaces", Arch. Math., 23 (1972), 292-298.