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"FIXED POINT THEOREM IN 2-NORMED SPACES"

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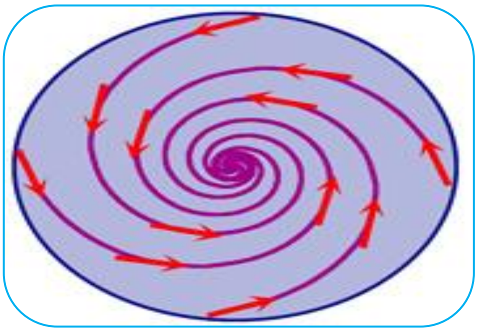
ABSTRACT:

This paper explores the fixed point theorem in 2-normed spaces, a generalization of traditional fixed point theorems applicable in metric spaces. We define the structure of 2-normed spaces, emphasizing the properties of the distance function that relies on pairs of vectors. We establish conditions under which a contraction mapping in a 2-normed space guarantees the existence and uniqueness of a fixed point. By employing iterative methods, we demonstrate convergence to the fixed point, showcasing the theorem's relevance in various fields, including functional analysis, economics, and computer science. The findings underscore the significance of the underlying space's geometry and the mapping's characteristics in the broader context of fixed point theory.



KEY WORDS: *Fixed point theorem, 2-normed spaces and pairs of vectors.*

INTRODUCTION

The concept of fixed points has profound implications across various branches of mathematics and its applications, serving as a cornerstone in areas such as analysis, topology, and applied mathematics. A fixed point of a function $T:V \to V$ is a point $x \in V$ such that $T(x)=x$. Understanding the conditions under which such points exist is crucial for solving equations and modeling dynamic systems.While classical fixed point theorems, such as the Banach fixed-point theorem, have established foundational results in metric spaces, the exploration of fixed points in more generalized settings has led to significant advancements. One such generalization is the framework of 2-normed spaces. Unlike traditional normed spaces, where distance is measured using a single vector, 2-normed spaces employ a function that evaluates distances between pairs of vectors, enriching the geometric structure and opening new avenues for analysis.

This paper aims to investigate the fixed point theorem in the context of 2-normed spaces, highlighting the unique characteristics of contraction mappings and their implications. We will explore the definitions, properties, and applications of 2-normed spaces, establishing the conditions under which a contraction guarantees the existence and uniqueness of a fixed point. Through iterative methods, we will illustrate the convergence to this fixed point, thereby demonstrating the theorem's utility in various mathematical and practical contexts.

A normed space is a vector space equipped with a function called norm. Geometrically, a norm is a tool to measurelength of a vector.

Definition 1. [E. Kreiszig. (1978). 5] Let X be a vector space with $\dim(X) \geq 2$. A mapping $||\cdot||: X \to \mathbb{R}$ that satisfies

(1) $\|x\| \geq 0$, for all $x \in X$;
$\|x\| = 0$ if and only if $x = 0$,
(2). $\|\alpha x\| = |\alpha| \|x\|$; for all $\alpha \in \mathbb{R}$ and $x \in X$,
(3). $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in X$

Is called a **norm**. A pair of $(X, \|\cdot,\cdot\|)$ is called a **normed space**.

In 1960's Gahler introduced a concept of n-normed spaces as a generalization of a concept of normed spaces. This space is equipped by an n-norm. The n-normed is used to measure volume of a parallelepiped spaned by n vectors. Especially for $n = 2$, the 2-norm is a tool to measure an area spaned by 2 vectors. T concept of 2-normed space was studied further by many researchers, [S. Ersan. (2019), M. Iranmanesh, F. Soleimany. (2016), A. Kundu. (2019) 1,4,6]. Now, we present some basic definition and properties of 2-normed spaces.

Definition 2. [R. W. Freese, Y. J. Cho. (2001) 2] Let X be a vector space with $\dim(X) \geq 2$. A mapping $||\cdot,\cdot||: X \to \mathbb{R}$ that satisfies

(N1). $\|x,y\| \geq 0$, for all $x,y \in X$;
$\|x,y\| = 0$ if and only if x,y linearly dependent,
(N2). $\|x,y\| = \|y,x\|$; for all $x,y \in X$,
(N3). $\|\alpha x,y\| = |\alpha| \|x,y\|$; for all $\alpha \in \mathbb{R}$ and $x,y \in X$,
(N4). $\|x+z,y\| \leq \|x,y\| + \|z,y\|$, for all $x, y, z \in X$

Is called a **2-norm**. A pair of $(X, \|\cdot,\cdot\|)$ is called a **2-normed space**.

Note that in 2-normed space$(X, \|\cdot,\cdot\|)$ we have
$\|x1,x2\| = \|x1, x2 + \alpha x1\|$, For all $\alpha \in \mathbb{R}$ and $x_1, x_2 \in X$.

Definition 3. [R. W. Freese, Y. J. Cho. (2001) 2] A sequence $\{x_k\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent if there is an $x \in X$ such that $\lim_{k\to\infty} \|x_k - x, z\| = 0$ for all $z \in X$.

If $\{x_k\}$ converges to x the we denote it by $x_k \to x$ as $k \to \infty$. The point x is called limit point of x_k.

Definition 4. [R. W. Freese, Y. J. Cho. (2001) 2] A sequence $\{x_k\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be a Cauchy sequence if there is an $x \in X$ such that $\lim_{k,l\to\infty} \|xk - x_l, z\| = 0$ for all $z \in X$.

Lemma 5. If A sequence $\{x_k\}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is convergent, then $\{x_k\}$ is a Cauchy sequence.

Definition 6. A 2-normed space is called complete if every Cauchy sequence is convergent.
Moreover, the complete 2-normed space is called a 2-Banach space.

Definition 7. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. A set $K \subset X$ is said to be closed if the limit point of every convergent sequence in K is also in K.

RESULTS AND DISCUSSION:

In this section, we define a normed derived from 2-norm and use this norm to prove a fixed point theorem in 2-normed space. We begin with defining the norm.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and $Y = \{y_1, y_2\}$ be a linearly independent set in X, we define a function in X by

$$\|x\| = \|x,y_1\| + \|x,y_2\| \qquad (1)$$

One can see that the function $\|\cdot\|: X \to \mathbb{R}$ defines in (1) defines a norm in X.

Theorem 8. $(X, \|\cdot\|)$ is a norm space, with $\|\cdot\|$ is a norm defined in (1).
Proof. We just need to prove that normed defined in (1) as a norm in X.

(1). By using (N1), one can see that for every $x \in X$ we have

$$\|x\| = \|x, y_1\| + \|x, y_2\| \geq 0,$$

Because each term on the above equation will greater or equals 0.

If $x = 0$, from then (N1) we have $\|x, y_1\| = 0$ and $\|x, y2\| = 0$,

which means $\|x\| = 0$.

If $\|x\| = 0$ then $\|x,y_1\| + \|x,y_2\| = 0$. Because each term is nonnegative then we shoud have $\|x,y_1\| = 0$ and $\|x,y_2\| = 0$. This means x is a vector that dependent only to $y1$ and also dependent only to y_2. The vector x must be 0.

(2). For any $x \in X$ and $\alpha \in \mathbb{R}$, $\|\alpha x\| = \|\alpha x, y_1\| + \|\alpha x, y_2\|$.
By using (N3) we have $\|\alpha x,y_1\| + \|\alpha x,y_2\| = |\alpha| (\|x,y_1\| + \|x,y_2\|) = |\alpha| \|x\|$. Then we have $\|\alpha x\| = |\alpha| \|x\|$; for all $\alpha \in \mathbb{R}$ and $x \in X$.

(3). For any $x, y \in X$ we have $\|x+y\| = \|x + y,y_1\| + \|x+y,y_2\|$. By using (N4) we also have $\|x + y, y_1\| + \|x + y, y_2\| \leq \|x,y_1\| + \|y,y_1\| + \|x,y_2\| + \|y,y_2\|$.
This means $\|x + y, y_1\| + \|x + y, y_2\| \leq \|x, y_1\| + \|x,y_2\| + \|y,y_1\| + \|y,y_2\| = \|x\|+\|y\|$. Hence, $\|x + y\| = \|x\| + \|y\|$.

We proved that the norm defined in (1) is a norm as desired then A pair of $(X,\|\cdot\|)$ is a normed space.

For simplicity, from now on we call the norm defined in (1) 'derived norm'. We will using this norm to prove a fixed point theorem in 2-normed space. Before that, we show in this following proposition a convergent sequence with respect to 2-norm also convergent with respect to derived norm.

The fixed point theorem in 2-normed spaces is a generalization of traditional fixed point theorems, such as the Banach fixed-point theorem, which typically applies to metric spaces. In a 2-normed space, the notion of distance is defined using two vectors, leading to a different structure.

2-Normed Spaces

A 2-normed space is a vector space V equipped with a function $\|\cdot,\cdot\|: V \times V \to R$ that satisfies certain properties:

1. **Non-negativity**: $\|x,y\| \geq 0$ for all $x, y \in V$ and $\|x,y\| = 0$ if and only if $x = y$.
2. **Symmetry**: $\|x,y\| = \|y,x\|$ for all $x, y \in V$.
3. **Triangle inequality**: $\|x,z\| \leq \|x,y\| + \|y,z\|$ for all $x, y, z \in V$.
4. **Homogeneity**: $\|\alpha x,y\| = |\alpha| \|x,y\|$ for any scalar α and all $x, y \in V$.

Fixed Point Theorems

In the context of 2-normed spaces, a fixed point theorem often asserts the existence of a fixed point for a continuous function under specific conditions. One common formulation involves:

1. Contractive Maps: A function $T: V \to V$ is said to be a contraction if there exists a constant $0 < k < 1$ such that $\|T(x),T(y)\| \leq k\|x,y\|$ for all $x, y \in V$.

2. Existence of Fixed Points: Under suitable conditions (e.g., compactness, completeness), one can prove that such a contraction has at least one fixed point xxx in V such that T(x) = x.

Applications:

These theorems are valuable in various fields, including:

- **Functional Analysis**: Finding solutions to integral and differential equations.
- **Economics**: Equilibrium analysis in game theory and optimization problems.
- **Computer Science**: Algorithms for fixed point computation in iterative methods.

CONCLUSION:

In this exploration of fixed point theorems within the framework of 2-normed spaces, we have established the conditions under which contraction mappings guarantee the existence and uniqueness of fixed points. By leveraging the unique properties of 2-normed spaces, we have shown how the interplay between the distance function defined on pairs of vectors and the contraction condition provides a rich structure for analysis. The findings affirm the relevance of fixed point theorems not only in theoretical mathematics but also in practical applications across various fields, including functional analysis, optimization, and computational algorithms. The ability to ensure convergence to a fixed point through iterative methods highlights the robustness of these results, making them invaluable tools for solving complex problems.

REFERENCES :

1. S. Ersan. (2019). Variations on ward continuity in 2-normed spaces. *AIP conference proceedings*. 2183, 050004.
2. R. W. Freese, Y. J. Cho. (2001). Geometry of Linear 2-Normed Space, Huntington N. Y.: Nova Publisher.
3. P. K. Harikrishnan, K. T. Ravindran. (2011). Some properties of accretive operators in linear 2-normed space. *International Mathematical Forum*, 3 (59), 2941-2947.
4. M. Iranmanesh, F. Soleimany. (2016). Some results on Farthest points in 2-normed spaces. *Novi Sad. J. Math*. 46(1), 2017-215.
5. E. Kreiszig. (1978). Introductory Functional Analysis with Applications, New York: John Willey and Sons.
6. A. Kundu. (2019). On metrizability and normability of 2-normed spaces. *Mathematical Sciences*, 13(1), 69-77.