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"STUDY OF COMMON FIXED-POINT THEOREMS IN METRIC SPACES AND THEIR APPLICATIONS"

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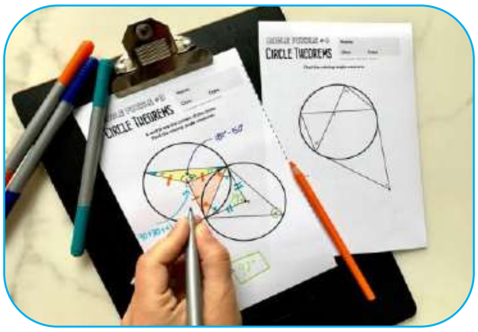
ABSTRACT:

The study of common fixed-point theorems and their applications encompasses a fundamental aspect of mathematical analysis with wide-ranging implications across various disciplines. Fixed point theorems provide a foundational understanding of points that remain unchanged under certain mappings, while common fixed-point theorems extend this concept to multiple mappings, elucidating conditions under which they share a fixed point. This abstract explores the theoretical underpinnings of common fixed-point theorems, their significance in mathematics, and their diverse applications in fields such as economics, computer science, and physics. By examining the conditions for existence and uniqueness of common fixed points and exploring their implications, these abstract aims to underscore the importance and versatility of these theorems in modeling and solving real-world problems across disciplines.



KEYWORDS: *Common fixed point, Theorems, Applications and Mathematical Analysis.*

INTRODUCTION

The study of common fixed point theorems and their applications lies at the intersection of mathematics and various disciplines such as economics, computer science, and physics. In mathematics, fixed point theorems establish conditions under which a function has a point that does not move when the function is applied. The concept of fixed points has applications in various mathematical areas, including analysis, topology, and algebra. Common fixed point theorems extend this concept to multiple functions or mappings, establishing conditions under which several functions have a shared fixed point. These theorems have applications in a wide range of fields, including optimization, game theory, and dynamical systems.

Understanding common fixed point theorems and their applications can lead to insights into the behavior of complex systems and provide tools for solving practical problems in diverse areas. For example, in economics, common fixed point theorems can be used to analyze equilibrium in markets or the existence of stable allocations in matching problems. In computer science, they can be applied to the design and analysis of algorithms for solving optimization problems or simulating dynamical systems.

Fixed point theorems concern maps f of a set X into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology.

Example - Suppose we are given a system of n equations in n unknowns of the form gj (x1, . . . , xn) = 0, j = 1, . . . , n where the gj are continuous real-valued functions of the real variables xj . Let hj (x1, . . . , xn) = gj (x1, . . . , xn) + xj , and for any point x = (x1, . . . , xn) define h(x) = (h1(x), . . . , hn(x)). Assume now that h has a fixed point $\bar{x} \in R^n$. Then it is easily seen that $\bar{x}$ is a solution to the system of equations. Various application of fixed point theorems will be given in the next chapter.

DISCUSSION:

Many authors have studied the fixed point theorem in metric space, general metric space, fuzzy metric space, probabilistic fuzzy metric space. One of such generalizations is generalized metric space (or D-metric space) initiated by Dhage in 1992. (Shaban Sedghi et, al. 2007) give new definition D* - metric space which is some modification of Dhage metric space (D-metric space). In this paper, we establish a common fixed point theorem for a class of weakly commuting mappings in complete D* - metric space.

Definition- Let X be a nonempty set. A generalized metric (or D* -metric) on X is a function, $D^*: X^3 \to [0,\infty)$ that satisfies the following conditions for each x, y, z ∈ X.

1. $D^*(x, y, z) \geq 0$,
2. $D^*(x, y, z) = 0$ if and only if $x = y = z$,
3. $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,
4. $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D*) is called a generalized metric (or D*-metric) space.

Examples-

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = R^n$ then we define

$$D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{1/p} \text{ for every } p \in R^+$$

(d) If X = R then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise}, \end{cases}$$

Remark- In a D*-metric space, we have $D^*(x, x, y) = D^*(x, y, y)$.

Proof: We have triangular inequality

(i) $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$ and similarly

(ii) $D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$.

Hence by (i), (ii) we get $D^*(x, x, y) = D^*(x, y, y)$.

Open ball with centre x and radius r : Let (X, D*) be a D*- metric space. For r > 0 define

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}$$

Common fixed-point theorems are fundamental results in mathematics that establish conditions under which two or more mappings have a fixed point in common. These theorems are crucial in various branches of mathematics and have applications in diverse fields including economics, computer science, and engineering. Here are some common fixed-point theorems and their applications:

1. Banach Fixed-Point Theorem (Contraction Mapping Theorem):
- **Statement:** If (X, d) is a complete metric space and T : X → X is a contraction mapping (i.e., there exists $0 \leq k < 1$ such that $d(Tx,Ty) \leq k \cdot d(x, y)$ for all x, y ∈ X x), then T has a unique fixed point in X.
- **Applications:** Used in functional analysis, optimization, numerical methods (e.g., iterative methods for solving equations), and in proving existence and uniqueness of solutions to differential equations.

2. Brouwer Fixed-Point Theorem:
- **Statement:** Every continuous function from a convex compact subset of Euclidean space to itself has at least one fixed point.
- **Applications:** Applied in topology, game theory (existence of Nash equilibria), and economics (existence of equilibrium points in economic models).

3. Kakutani Fixed-Point Theorem:
- **Statement:** If X is a non-empty compact convex subset of a locally convex Hausdorff topological vector space, and $f : X \to 2^X$ (where 2^X denotes the power set of X) is an upper semi-continuous correspondence with non-empty compact convex values, then f has a fixed point.
- **Applications:** Used in game theory (proof of existence of Nash equilibria in games with continuous payoffs) and in economics (existence of competitive equilibrium in markets with continuous utility functions).

4. Schauder Fixed-Point Theorem:
- **Statement:** If X is a Banach space and f : X → X is a completely continuous map (compact map), then f has a fixed point.
- **Applications:** Used in proving the existence of solutions to integral equations, in topology (fixed-point results in Banach spaces), and in functional analysis.

5. Knaster-Tarski Fixed-Point Theorem:
- **Statement:** For a complete lattice (L, ≤), every order-preserving function f : L → L has at least one fixed point.
- **Applications:** Used in lattice theory, logic (in the study of fixed-point logics), and in computer science (in the study of fixed-point combinatorics and fixed-point theorems in domain theory).

These theorems and their variations play a crucial role in theoretical mathematics and have numerous practical applications across various disciplines, providing powerful tools to prove existence and uniqueness results for solutions to equations and problems involving mappings.

CONCLUSION:
In conclusion, fixed-point theorems are foundational results in mathematics with broad applications across various disciplines. They provide powerful tools for proving the existence and sometimes uniqueness of solutions to equations and mappings in different contexts. Key theorems like the Banach, Brouwer, Kakutani, Schauder, Poincaré-Birkhoff, and Knaster-Tarski theorems establish conditions under which fixed points exist, often in spaces ranging from metric spaces to Banach spaces, and even in more abstract settings like lattices or topological spaces.

These theorems not only have theoretical significance but also practical implications in fields such as economics, game theory, optimization, dynamical systems, and computer science. They enable researchers and practitioners to establish the existence of equilibrium points, stable solutions, or optimal configurations in various real-world and theoretical problems. Overall, fixed-point theorems stand as fundamental pillars of mathematical analysis, offering robust frameworks for understanding and solving problems across diverse domains.

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