REVIEW OF RESEARCH
ISSN: 2249-894X
IMPACT FACTOR : 5.7631(UIF)
UGC APPROVED JOURNAL NO. 48514
VOLUME - 8 | ISSUE - 9 | JUNE - 2019

A STUDY OF APPLICATIONS OF H-FUNCTION IN GENERAL STRUCTURE OF GENERALIZED GAMMA DENSITY

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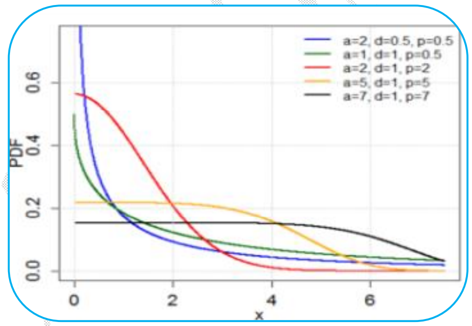
ABSTRACT:
In the present paper, the author has studied about the structures which are the products and ratios of statistically independently distributed positive real scalar random variables.The author has derived the exact density of Generalized gamma density by the Hamkel Transform of the unknown density and afterthat the unknown density has been derived in terms of certain generalized hypergeometric functions by taking the Inverse Hankel Transform .A more general structure of generallized Gamma density has also been discussed.



KEYWORDS: : *Generalized Gamma Density, H -function, Hankel Transform, Inverse Hankel Transform. (2010 Mathematics Subject Classification: 33CXX, 44A15, 82XX)*

1. INTRODUCTION

General structures

A real scalar random variable x is said to have a real generalized gamma density iwhen the density is of the form:

$$f(x)=\begin{cases}\dfrac{\beta a^{\frac{\alpha}{\beta}}}{\Gamma\left(\frac{\alpha}{\beta}\right)}x^{\alpha-1}e^{-ax^{\alpha}}\\ 0, elsewhere\end{cases} \quad (1.1)$$

For $x>0,\alpha>0,\beta>0,a>0$.

Where the parameters α and β are real. The following discussion holds even when α and β are complex quantities. In this case, the conditions become $\operatorname{Re}(\alpha)>0,\operatorname{Re}(\beta)>0$ where Re(.)means the real part of (.).

Consider a set of real scalar random variables $x_1,\ldots,x_k$, statistically independently distributed, where x_j has the density in (1.1) with the parameters $\alpha_j,\beta_j; j=1,\ldots,k$ and consider the product

$$u=x_1x_2\ldots x_k \quad (1.2)$$

In the standard terminology in statistical literature, the h^{th} moment of u, when u has the density in (1.1), is given by

$$E\left(x^h\right)=\frac{\Gamma\left(\dfrac{\alpha+h}{\beta}\right)}{\Gamma\left(\dfrac{\alpha}{\beta}\right)a^{\frac{h}{\beta}}},\ for\ \operatorname{Re}(\alpha+h)>0 \quad (1.3)$$

Due to statistical independence,

$$E\left(u^h\right)=\left[E(x_1^h)\right]\left[E(x_2^h)\right]\ldots\left[E(x_k^h)\right]$$

$$= \prod_{j=1}^{k} \frac{\Gamma\left(\frac{\alpha_j + h}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a^{\frac{h}{\beta_j}}}, \ for\, \mathrm{Re}(\alpha + h) > 0,\ j = 1,\ldots,k \qquad (1.4)$$

The H -function is defined by means of a Mellin-Barnes type integral in the following manner (Mathai and Saxena, 1978):

$$H(z) = H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \Big|_{(b_q, B_q)}^{(a_p, A_p)} \right]$$

$$= H_{p,q}^{m,n}\left[z \Big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)} \right] = \frac{1}{2\pi i} \int_L \theta(s) z^{-s} ds \qquad (1.5)$$

Where $i = \sqrt{-1},\ z \neq 0$ and $z^{-s} = \exp[-\sin|z| + i \arg z]$ where $|z|$ represents the natural logarithim of $|z|$ and $\arg z$ is not the principal value. Here

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{j=1}^{n} \Gamma\left(1 - a_j - A_j s\right)}{\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j s\right) \prod_{j=n+1}^{p} \Gamma\left(a_j + A_j s\right)} \qquad (1.6)$$

The Hankel transform of $f(x)$ denoted by $H_v\{f(x); p\}$ or $F_v(p)$ is given by

$$H_v\{f(x); p\} = \int_0^{\infty} x J_v(px) f(x) dx \qquad (1.7)$$

2. Hankel Transform of $g(u)$

We can calculate the Hankel transform of $g(u)$ of u from the property of the statistical independent and is given by:

$$E[uJ_v(pu)] = E\left[x_1 J_v(px_1)\right] E\left[x_2 J_v(px_2)\right] \ldots E\left[x_k J_v(px_k)\right] \qquad (2.1)$$

$$E\left[uJ_v(pu)\right] = \int_0^{\infty} uJ_v(pu) g(u) du \qquad (2.2)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(v + r + 1} \left(\frac{p}{2}\right)^{v+2r} \int_0^{\infty} u^{v+2r+1} g(u) du$$

$$= J_v(p)\int_0^\infty u^{v+2r+1} g(u)du \qquad (2.3)$$

Now taking the Mellin transform of $g(u)$ with h replaced by $(v+2r+2)-1$.

$$E\left(u^{v+2r+1}\right) = J_v(p)\prod_{j=1}^{k} \frac{\Gamma\left(\frac{\alpha_j+v+2r+1}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a_j^{\frac{v+2r+1}{\beta_j}}} \qquad (2.4)$$

The unknown density $g(u)$ is obtained in terms of H-function by taking the inverse Hankel transform of (2.4). That is

$$g(u) = J_v(p)\prod_{j=1}^{k} \frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)} H_{0,k}^{k,0}\left[u\left|\begin{matrix} --- \\ \left(\frac{\alpha_j}{\beta_j},\frac{1}{\beta_j}\right)\end{matrix}\right.; j=1,\ldots,k\right] \qquad (2.5)$$

Where $s = v+2r+2$ and $s > 0$.

If we take $\beta_j = 1; j = 1,\ldots,k$, the H-function reduces to the G-function, for special values of k, one can evaluate (2.5) in terms of elementry special functions.

If we consider more general structures in the same category. For example, consider the structure

$$u_1 = x_1^{\gamma_1} x_2^{\gamma_2} \ldots x_k^{\gamma_k}, \gamma_k > 0, j = 1,\ldots,k \qquad (2.6)$$

Where $x_1,\ldots,x_k$ are mutually independently distributed as in (2.1).

Then the Hankel transform of $g(u_1)$ of u_1 is obtained from the property of the statistical independent and is given by:

$$E[u_1 J_v(pu_1)] = E\left[x_1^{\gamma_1} J_v(px_1^{\gamma_1})\right] E\left[x_2^{\gamma_1} J_v(px_2^{\gamma_2})\right] \ldots E\left[x_k^{\gamma_1} J_v(px_k^{\gamma_k})\right]$$

$$= J_v(p)\prod_{j=1}^{k} \frac{\Gamma\left(\frac{\alpha_j+\gamma_j(v+2r+1)}{\beta_j}\right)}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right) a_j^{\frac{(v+2r+1)\gamma_j}{\beta_j}}} \qquad (2.7)$$

The unknown density $g(u_1)$ is obtained in terms of H-function by taking the inverse Hankel transform of (2.7). That is

$$g(u_1) = J_v(p)\prod_{j=1}^{k}\frac{a_j^{\frac{\gamma_j}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)} H_{0,k}^{k,0}\left[u\left|_{\left(\frac{\alpha_j}{\beta_j},\frac{\gamma_j}{\beta_j}\right)}^{---}; j=1,\ldots,k\right.\right] \tag{2.8}$$

Where $s = v + 2r + 2$ and $s > 0$.

3. A More General Structure

We can consider more general structures. Let

$$w = \frac{x_1, x_2, \ldots, x_r}{x_{r+1}, \ldots, x_k} \tag{3.1}$$

Where $x_1, \ldots, x_k$, mutually independently distributed real random variables having the density in (1.1) with x_j having parameters $\alpha_j, \beta_j; j = 1, \ldots, k$.

Then the Hankel transform of $g(w)$ is given as:

$$E[wJ_v(pw)] = E\left[x_1 J_v(px_1)\right]\ldots E\left[x_r J_v(px_r)\right]$$
$$E\left[x_{r+1}^{-1} J_v\left(px_{r+1}^{-1}\right)\right]\ldots E\left[x_k^{-1} J_v\left(px_k^{-1}\right)\right] \tag{3.2}$$

$$= J_v(p)\left\{\prod_{j=1}^{r}\frac{1}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)a_j^{\frac{2r+1}{\beta_j}}}\right\}\left\{\prod_{j=r+1}^{k}\frac{1}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)a_j^{\frac{-(2r+1)}{\beta_j}}}\right\}$$
$$\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{\alpha_j+s-1}{\beta_j}\right)}{a_j^{\frac{s-1}{\beta_j}}}\right\}\left\{\prod_{j=r+1}^{k}\frac{\Gamma\left(\frac{\alpha_j-s+1}{\beta_j}\right)}{a_j^{\frac{1-s}{\beta_j}}}\right\} \tag{3.3}$$

The unknown density $f(x)$ is obtained in terms of H-function by taking the inverse Hankel transform of (3.3). That is

$$g(w)=J_v(p)\left\{\prod_{j=1}^{r}\frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}\left\{\prod_{j=r+1}^{k}\frac{a_j^{-\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}$$

$$H_{k-r,k}^{r,k-r}\left[w\frac{\prod_{j=1}^{r}a_j^{\frac{1}{\beta_j}}}{\prod_{j=r+1}^{k}a_j^{-\frac{1}{\beta_j}}}\left|\begin{matrix}\left(1-\frac{\alpha_j-1}{\beta_j}\right);j=1,\ldots,r\\ \left(\frac{\alpha_j-1}{\beta_j},\frac{1}{\beta_j}\right);j=r+1,\ldots,k\end{matrix}\right.\right] \tag{3.4}$$

For $\beta_j=1; j=1,\ldots,k$, the H -function reduces to the G -function.

If we consider more general structures in the same category. Let, consider the structure

$$w_1=\frac{x_1^{\gamma_1},\ldots,x_r^{\gamma_r}}{x_{r+1}^{\gamma_{r+1}},\ldots,x_k^{k}} \tag{3.5}$$

Where $x_1,\ldots,x_k$, mutually independently distributed real random variables having the density in (3.5) with x_j having parameters $\alpha_j,\beta_j; j=1,\ldots,k$.

Then the Hankel transform of $g(w_1)$ is given as:

$$E[w_1(pw_1)]=E\left[x_1^{\gamma_1}J_v(px_1^{\gamma_1})\right]\ldots E\left[x_r^{\gamma_r}J_v(px_r^{\gamma_r})\right]$$
$$E\left[x_{r+1}^{-\gamma_{r+1}}J_v\left(px_{r+1}^{-\gamma_{r+1}}\right)\right]\ldots E\left[x_k^{-\gamma_k}J_v\left(px_k^{-\gamma_k}\right)\right] \tag{3.6}$$

$$=J_v(p)\left\{\prod_{j=1}^{r}\frac{1}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)a_j^{\frac{2r+1}{\beta_j}}}\right\}\left\{\prod_{j=r+1}^{k}\frac{1}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)a_j^{\frac{-(2r+1)}{\beta_j}}}\right\}$$

$$\left\{\prod_{j=1}^{r}\frac{\Gamma\left(\frac{\alpha_j+(s-1)\gamma_j}{\beta_j}\right)}{a_j^{\frac{(s-1)_j}{\beta_j}}}\right\}\left\{\prod_{j=r+1}^{k}\frac{\Gamma\left(\frac{\alpha_j-(s-1)\gamma_j}{\beta_j}\right)}{a_j^{\frac{(1-s)\gamma_j}{\beta_j}}}\right\} \tag{3.7}$$

The unknown density $g(w_1)$ is obtained in terms of H -function by taking the inverse Hankel transform of (3.7). That is

$$g(w_1) = J_v(p)\left\{\prod_{j=1}^{r}\frac{a_j^{\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}\left\{\prod_{j=r+1}^{k}\frac{a_j^{-\frac{1}{\beta_j}}}{\Gamma\left(\frac{\alpha_j}{\beta_j}\right)}\right\}$$

$$H_{k-r,k}^{r,k-r}\left[w\frac{\prod_{j=1}^{r}a_j^{\frac{1}{\beta_j}}}{\prod_{j=r+1}^{k}a_j^{-\frac{1}{\beta_j}}}\left|\begin{matrix}\left(1-\frac{\alpha_j-1}{\beta_j},\frac{\gamma_j}{\beta_j}\right);j=1,\ldots,r\\ \left(\frac{\alpha_j-1}{\beta_j},\frac{\gamma_j}{\beta_j}\right);j=r+1,\ldots,k\end{matrix}\right.\right] \qquad (3.8)$$

For $\beta_j = 1, \gamma_j = 1; j = 1,\ldots,k$, the H -function reduces to the G -function.

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