REVIEW OF RESEARCH

A BASIS EXAMINATION METHOD FOR MULTILEVEL LINEAR/LINEAR FRACTIONAL PROGRAMMING PROBLEM

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ABSTRACT:

In this study, we propose a basis examination method to find the global optima of the multi-level linear/ linear fractional programming problem in which the objective function of the first level is linear and the objective functions of other levels are linear fractional. The feasible region is a polyhedron. Here we prove that the global optima is an extreme point of the polyhedron. This is proved by taking into account the relationship between feasible solutions of the problem and basis of the technological coefficient submatrix associated to the variables of a particular level. A numerical example demonstrates the feasibility of the proposed approach.

KEYWORDS: *Global optima, polyhedron, technological coefficient submatrix .*

INTRODUCTION:

Multi-level programming (MLP) is an important and challenging branch of mathematical programming. It has applications in modelling and solving decentralised planning problems in which more than one decision-maker seeks their own interest. Bi-level programming (BLP) is a special case of MLP with a structure of two-levels in a hierarchical decision system. An important characteristic of MLP problems is that a planner at a certain level of hierarchy may have his objective function whereas the decision space is determined, partially, by other levels (Anandalingam, 1988; Vicente and Calamai, 1994; Wen and Hsu, 1991).

Bi-level linear programming has been introduced by Candler and Townsley (1982). Also, a bibliography of references on BLP and MLP in both linear and nonlinear cases has been given and a survey on fractional programming has been done which covers applications as well as major algorithmic and theoretical developments (Vicente and Calamai, 1994; Schaible, 1995; Colson, Marcotte and Savard, 2007). In case of bilevel programming problems, it has been proved that there exists an extreme point of the polyhedron that gives the optimal solution to the problem (Calvete and Gale, 1998, 1999, 2004). The maximum of a weighted sum of the objective functions in a multiple objective linear fractional programming (MOLFP) can be computed (Costa, 2007). The linear fractional programming problems can be solved by an iterative method based on the conjugate gradient projection method (Tantawy, 2008). A global optimal solution of multi-level linear fractional programming problems has already been given for integer case (Bhargava, 2011).

In this paper, we prove that under the usual assumptions there is an extreme point of the polyhedron S which solves the given problem. Also, we propose an enumerative method to efficiently search for a basis which provides an optimal solution. One of its most outstanding features is that this implicit search is only made among basis of the technological coefficient sub-matrix corresponding to variables of each level one by one.

The paper is organized as follows: In next section multi-level linear/ linear fractional programming problem is defined which is followed by some assumptions and notations. After that some preliminaries, two theorems and the algorithm are given. In addition, to facilitate the comprehension of the algorithm a numerical example is solved. Finally, conclusions are drawn.

PROBLEM FORMULATION

The multi-level linear/ linear fractional programming problem (MLLFP) is defined as:

$$\underset{x_1}{Min.}\, f_1 = k^1 x_1 + k^2 x_2 + \dots k^n x_n$$

$$where (x_2, x_3, \dots x_n) solves$$

$$\underset{x_2}{Min.}\, f_2 = \frac{(\alpha_1 + c_1^{11} x_1 + c_1^{12} x_2 + \dots c_1^{1n} x_n)}{(\beta_1 + c_1^{21} x_1 + c_1^{22} x_2 + \dots c_1^{2n} x_n)}$$

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$$where (x_n) solve$$

$$\underset{x_n}{Min.}\, f_n = \frac{(\alpha_{n-1} + c_{n-1}^{11} x_1 + c_{n-1}^{12} x_2 + \dots c_{n-1}^{1n} x_n)}{(\beta_{n-1} + c_{n-1}^{21} x_1 + c_{n-1}^{22} x_2 + \dots c_{n-1}^{2n} x_n)} \quad \text{s.t } (x_1, x_2, x_3, \dots x_n) \in S$$

Where $x_1 \in R_1^n, x_2 \in R_2^n, \dots x_n \in R_n^n$ are the variables controlled by the first, second and the n[th] level decision maker respectively. $k^1, k^2 \dots k^n, c_i^{11}, c_i^{12} \dots c_i^{1n}, c_i^{21} c_i^{22} \dots c_i^{2n}, \forall i = 1, 2 \dots n-1$ are the vectors of conformal dimension, α_i and β_i are scalars $\forall i = 1, 2 \dots n-1$ and the common constraint region to all the levels is a polyhedron i.e $S = \{(x_1, x_2 \dots x_n) : A^1 x_1 + A^2 x_2 + \dots A^n x_n = b, x_j \geq 0, \forall j = 1, 2 \dots n\}$ where A^j is a m×n matrix and $\forall j = 1, 2 \dots n$,b is an m-vector.

Some Assumptions and Notations

Let us introduce some additional assumptions and notations.

- We assume that polyhedron S is non-empty and compact.
- Matrix A^n has full row rank and $m < n_n$.
- $\beta_i + c_i^{21} x_1 + c_i^{22} x_2 + \dots c_i^{2n} x_n > 0$, $\forall (x_1, x_2, \dots x_n) \in S$ otherwise it is sufficient to consider the feasible region of the first level as $S_1 = \{x_1 \in R_1^n : (x_1, x_2 \dots x_n) \in S\}$. Let S_1 be the projection of S onto R_1^n i.e and $S_1 = \{x_1 \in R_1^n : (x_1, x_2 \dots x_n) \in S\}$.
- $V_1, V_2 \dots V_n$, are the sets of indices of first, second and so on up till n[th] level controlled variables respectively.
- The feasible region of the second level decision maker is $S_2(x_1) = \{x_2 \in R_2^n : A^2 x_2 + A^3 x_3 + \dots A^n x_n \leq b - A^1 x_1, x_2 \geq 0\}$.

- The feasible region of the n^{th} level decision maker is $S_n\left(x_1,x_2...x_{n-1}\right)=\left\{x_n\in R_n^n : A^n x_n\le b-A^1x_1-A^2x_2-A^3x_3+...A^{n-1}x_{n-1},x_{n-1}\ge 0\right\}$.
- The inducible region of the first level decision maker will be denoted by $\left(IR\right)=\left\{\left(x_1,x_2,...x_n\right): x_i\ge 0,\forall i=1,2...n-1\right\}$
$$x_n=\arg\min.\left\{\frac{(\alpha_{n-1}+c_{n-1}^{11}x_1+c_{n-1}^{12}x_2+...c_1^{1n}x_n)}{(\beta_{n-1}+c_{n-1}^{21}x_1+c_{n-1}^{22}x_2+...c_{n-1}^{2n}x_n)}: A^1x_1+A^2x_2+...A^nx_n=b,x_n\ge 0\right\}.$$
- Thus for each value of the first level variables, there will be a unique solution to the second level problem and likewise for each value of the (n-1)th level variable x_{n-1} $\exists$ a unique solution to the n^{th} level problem x_n.

Theorem 1

The inducible region of MLLFP is formed by the union of connected faces of S and an optimal solution to MLLFP occurs at an extreme point of polyhedron S.

Proof:

Notice that the first level objective function is linear. Hence it is both convex and concave. Since it is also differentiable, then it is in particular quasiconcave. On the other hand, the second third and so on up till n^{th} level objective functions are the ratio of two affine functions. Hence they are quasi-concave. Now the first level decision maker minimizes a continuous function over a compact set. Hence there exists a minimizing solution to the MLLFP. By the definition above, it is clear that the induced region is formed by the connected faces of S. thus if $IR=\bigcup_j S_j$ $\exists$ at least one j which gives a minimizing solution to the MLLFP and as each S_j is a nonempty compact polyhedron there exist an extreme point of polyhedron S_j and hence of S so that the optimal solution of MLLFP occurs at that point. (Refer Theorem 1, Calvate and Gale; 1999)

Preliminaries

Here we develop the algorithm for solving multi-level linear/linear fractional programming problems based on basis examination method. The idea of basis examination was proposed in (Candler & Townsley, 1982). Earlier (Calvate & Gale, 1999) applied this method to solve bi-level linear/linear fractional programming problems. Here we are imposing more restrictions on the examinations of basis which are the sub-matrices of A^i which lead to the global optimal solution to the problem MLLFP. We also find the necessary conditions which when imposed are giving better solutions and preventing us from returning back to any formerly analyzed basis.

At first for each $x_1 \in S_1$, a feasible solution to the second level problem is obtained by solving the following linear fractional problem.

$$\underset{x_2}{Min.}\, f_2=\frac{\left(\alpha_1^{'}+c_1^{12}x_2+...c_1^{1n}x_n\right)}{\left(\beta_2^{'}+c_2^{23}x_3+...c_2^{2n}x_n\right)}$$

s.t. $x_2\in S\left(x_1\right)$

Where $\alpha_1' = \alpha_1 + c_1^{11}x_1$, $\beta_1' = \beta_1 + c_1^{21}x_1$

Hence an extreme point of the polyhedron $S_2(x_1)$ is obtained. Then for each (x_1, x_2) we obtain a feasible solution to the third level problem by solving the following linear fractional problem.

$$\underset{x_3}{Min.}\, f_3 = \frac{\left(\alpha_2' + c_2^{13}x_3 + \dots c_2^{1n}x_n\right)}{\left(\beta_2' + c_2^{23}x_3 + \dots c_2^{2n}x_n\right)} \quad \text{s.t. } x_3 \in S_3(x_1, x_2)$$

where $\alpha_2' = \alpha_2 + c_1^{11}x_1 + c_1^{12}x_2 \quad \beta_2' = \beta_2 + c_1^{21}x_1 + c_2^{22}x_2$

$$\underset{x_n}{Min.}\, f_n = \frac{\left(\alpha_{n-1}' + c_n^{1n}x_n\right)}{\left(\beta_{n-1}' + c_n^{2n}X_n\right)} \quad \text{s.t. } x_n \in S_n(x_1, x_2 \dots x_{n-1})$$

Thus first an extreme point is obtained and the point so obtained $(x_1, x_2 \dots x_{n-1})$ thus belongs to the inducible region (IR). Then we consider a basis B_i of A^i and check the optimality conditions.. To solve the fractional problems, parametric approach is considered. In this case, it is known that an optimal solution to the following linear parametric problem verifying $F_{i-1}(\lambda_{i-1}) = 0$ is an optimal solution to

LP$(x_1, x_2 \dots x_i)$: $F_i(\lambda_i) = \left(\alpha_i' + c_i^{1i+1}x_{i+1} + \dots c_i^{1n}x_n\right) - \lambda_i\left(\beta_i' + c_i^{2i+1}x_{i+1} + \dots c_i^{2n}x_n\right)$ $x_{i+1} \in S_{i+1}(x_1, x_2 \dots x_i)$

Here first we have to show that $\exists (x_1, x_2 \dots x_i)$ s.t B_{i-1} is a basis to a feasible solution of LP $(x_1, x_2 \dots x_i)$. Secondly we have to test that B_{i-1} verifies the optimality conditions of problem LP $(x_1, x_2 \dots x_i)$ for some value of parameter λ_{i-1} and finally, that for one of these values $F_{i-1}(\lambda_{i-1}) = 0$. Regarding the optimality conditions of problem LP$(x_1, x_2 \dots x_i)$, it suffices to check that the following reduced costs are greater than or equal to zero, regardless of the existence of $(x_1, x_2 \dots x_i)$,

$$\left(c_{i-1j}^{1i} - \lambda_{i-1}c_{i-1j}^{2i}\right) - \left(c_{i-1B}^{1i} - \lambda_{i-1}c_{i-1B}^{2i}\right)B_{i-1}^{-1}A_j^i \geq 0 \forall j \qquad \dots(1)$$

Where c_{i-1j}^{1i} and c_{i-1j}^{2i} are the j[th] component of vectors c_{i-1j}^{1i} and c_{i-1j}^{2i} respectively and c_{i-1B}^{1i} and c_{i-1B}^{1i} are the m-row vectors of c_{i-1}^{1i} and c_{i-1}^{2i} associated to the basic variables of B_{i-1} and A_j^i is the j[th] column of A^i. Let $\left[\lambda_{i-1}^1, \lambda_{i-1}^u\right]$ be the interval of parameter λ_{i-1}, computed by setting condition (1). If $\lambda_{i-1}^1 = -\infty$ or $\lambda_{i-1}^u = \infty$ then the interval $\left[\lambda_{i-1}^1, \lambda_{i-1}^u\right]$ will be open in that extreme.

If there exists no value of λ_{i-1} such that basis B_{i-1} verifies condition (1), then this basis is of no interest because it is impossible to obtain a point of the inducible region corresponding to it. Otherwise, we must also ask for the existence of a $\lambda_{i-1} \in \left[\lambda_{i-1}^1, \lambda_{i-1}^u\right]$ such that $F_{i-1}(\lambda_{i-1}) = 0$ and $x_{i+1} \in S_{i+1}(x_1, x_2 \dots x_i)$ such that B_{i-1} is a feasible basis to this problem. Thus we can establish the following subset of (IR), which corresponds to each basis B_{i-1} from A^i verifying condition (1). $\{(x_1, x_2 \dots x_i) : x_1, \dots x_i \geq 0\}, x_i = (B_{i-1}^{-1}(b - A^1 x_1 - \dots A^{i-1} x_{i-1}), 0)$, $B_{i-1}^{-1}(b - A^1 x_1 - \dots A^{i-1} x_{i-1}) \geq 0,$

$$\lambda_{i-1}^l \leq \left[\frac{\alpha_{i-1} + c_{i-1}^{11} x_1 + \dots\dots c_{i-1}^{1i-1} x_{i-1} + c_{iB}^{1i} B_i^{-1}(b - A^1 x_1 - \dots\dots A^i x_i)}{\beta_{i-1} + c_{i-1}^{21} x_1 + \dots\dots c_{i-1}^{2i-1} x_{i-1} + c_{iB}^{1i} B_i^{-1}(b - A^1 x_1 - \dots\dots A^i x_i)} \right] \leq \lambda_{i-1}^u$$

Therefore, if this set is non empty, the best point of the inducible region corresponding to basis B_{i-1} is obtained by solving the following linear problem.

$$P(B_{i-1}) = Min.\ k^1 x_1 + \dots k^{i-1} x_{i-1} + k_{Bi-1}^i x_{iB} + \dots k^n x_n$$

s.t. $A^1 x_1 + \dots A^{i-1} x_{i-1} + B_{i-1} x_{iBi-1} + \dots A^n x_n = b$

$$(\lambda_{i-1}^1 c_{i-1}^{21} - c_{i-1}^{11}) x_1 + (\lambda_{i-1}^1 c_{i-1}^{22} - c_{i-1}^{12}) x_2 + \dots (\lambda_{i-1}^1 c_{iBi-1}^{2i} - c_{i-1Bi-1}^{1i}) x_{iBi-1} \leq \alpha_{i-1} - \lambda_{i-1}^1 \beta \qquad \dots(2)$$

$$(c_{i-1}^{11} - \lambda_{i-1}^u c_{i-1}^{21}) x_1 + (c_{i-1}^{12} - \lambda_{i-1}^u c_{i-1}^{22}) x_2 + \dots (c_{i-1Bi-1} - \lambda_{i-1}^u c_{i-1Bi-1}^{2i}) x_{iBi-1} \leq \lambda_{i-1}^u \beta_{i-1} - \alpha_{i-1} \qquad \dots(3)$$

$$x_1, x_2 \dots x_{iBi-1} \geq 0$$

Where x_{iBi-1} stands for the variables x_i of related to basis B_{i-1} and k_{Bi-1}^i is the mth row vector of k^i associated to these variables. Notice that, while basis B_{i-1} is being analyzed, the variables of the ith level not associated to it remain equal to zero. We also introduce the following relaxed problem, which does not take into account constraints (2) and (3).

$$P_R(B_{i-1}) : Min. : k^1 x_1 + \dots + k_{Bi-1}^i x_{iBi-1} + \dots k^n x_n$$

$$A^1 x_1 + \dots A^{i-1} x_{i-1} + B_{i-1} x_{iBi-1} = b, \quad x_1, x_2 \dots x_{iBi-1} \geq 0$$

Now we are extending the result given by Calvate and Gale (1999) for multi-level linear/linear fractional programming problems. (Refer lemmas 1-4, Calvate and Gale; 1999)

Theorem 2.

Problems $P(B_{i-1})$ and $P_R(B_{i-1})$ are equivalent and to obtain a better point of induced region IR , basis A^i should include at least one vector with indices $j \in R$ and z_j <0. Also basis A^i will be feasible only if it has at least one vector whose index $j \in V_i - T_{i-1}$ and $z_j' < 0$.

Proof:

At first to prove that problems $P(B_{i-1})$ and $P_R(B_{i-1})$ are equivalent, we have to prove that they have the same optimal solution. As for the optimal solution, difference in the objective functions of $P(B_{i-1})$ and $P_R(B_{i-1})$ are due to basic variables which are slack variables and have zero cost coefficient, the value of the objective function will remain same for the optimal solution and if none of the optimal solution of problem $P_R(B_{i-1})$ verify the constraints (2) and (3) $P_R(B_{i-1})$ and hence $P(B_{i-1})$ is not feasible. Hence problems $P(B_{i-1})$ and $P_R(B_{i-1})$ are equivalent.

Now to obtain better point of IR let $x' = \left(x_1', x_2' \ldots x_i'\right)$ be the optimal solution of $P(B_{i-1})$. To proceed we divide the set of indices to two parts, one associated to basis B_{i-1} and the other associated to non-basic variables and denote them by T_{i-1} & B_{i-1} respectively. Let R denote the set of indices of non basic variables corresponding to x'.

Let $f_1(x')$ be the value of the first-level objective function at x′. Then $f_1(x) = f_1(x') + \sum_{j\in R} z_j x_j$, $z_j = k_j - k_Q Q^{-1} A_j^i$. where k_j is the j[th] cost coefficient in f_1, k_Q is the m- row vector of $k = \left[k^1, k^2 \ldots k^n\right]$ associated to basic variables of Q and A_j^i denotes the j[th] column vector of matrix $\left[A^1, A^2 \ldots A^n\right]$ at the i[th] level. For optimality, now if the matrix $\left[A^1, A^2 \ldots A^n\right]$ is decomposed to $[Q,N]$ here Q and N are the matrices associated to basic and non-basic variables respectively. Thus to obtain a better point of induced region IR, basis A^i should include at least one vector with indices $j \in R$ and z_j <0. If there is no basis current best point of IR is global optimum to the problem. Now for A^i to be feasible for problem $P_R(B_{i-1})$, artificial variables remaining in the optimal basis of phase 1 are removed by placing variables x_j, $j \in V_1 \cup V_2 \ldots \cup V_i$ with z_j' in the basis. As Phase 1 has been concluded and $z_j' \geq 0 \ \forall$ $j \in V_1 \cup V_2 \ldots \cup V_{i-1}$, basis A^i will be feasible only if it has at least one vector whose index $j \in V_i - T_{i-1}$ and $z_j' < 0$.

Now if we have previously built sets $C_1^1, C_1^2 \ldots C_1^i$,let E_1 denote the set of all sets C_1. If we have previously built sets $C_2^1, C_2^2 \ldots C_2^i$, let E_2 denote the set of all sets C_2. Likewise E_3 is formed. Thus E_1, E_2, E_3 are sets of indices. Then, in order to select a set of vectors $B_{i-1}^1, B_{i-1}^2, \ldots B_{i-1}^j$ of A^i that can form a basis of interest, we will have to guarantee that the sets of indices of these vectors includes at least one index from each element of E_1, at least one index from each element of E_2, and not all indices from each element of E_3. Hence, to find this set of indices we suggest solving in w_j the following system.

P2: $$\sum_j w_j \delta_j \geq 1, \delta_j = \begin{cases} 1, j \in C_1 \\ 0, otherwise \end{cases}$$

$$C_1 \in E_1$$

$$\sum_j w_j \delta_j \geq 1, \delta_j = \begin{cases} 1, j \in C_2 \\ 0, otherwise \end{cases}$$

$$C_2 \in E_2$$

$$\sum_j w_j \delta_j \leq \sum_j \delta_j - 1, \delta_j = \begin{cases} 1, j \in C_3 \\ 0, otherwise \end{cases}$$

$C_3 \in E_3$

$\sum_j w_j = m$

$w_j \in \{0,1\},\ j \in V_i$

The required set will be constituted by vectors of A^i with indices j such that the corresponding w_j =1.

On the other hand, note that the following linear problem provides a lower bound on the objective function of problem MLLFP.

$P_R:\ Min.\ k^1 x_1 + k^2 x_2 + \ldots k^n x_n$ s.t. $(x_1, x_2 \ldots x_n) \in S$

Therefore, if in some step of the algorithm a point of the inducible region is found whose objective function value equals the optimum of P_R, then this point is a global optimum of MLLFP. However, in general, the lower bound provided by P_R will be very far from the optimum of MLLFP.

The Algorithm

Step 0

i. Solve problem P_R.
ii. If P_R is not feasible neither is MLLFP, Stop.
iii. Set i=i+1
iv. Let $(x_1^{'}, x_2^{'} \ldots x_n^{'})$ be an optimal solution of P_R. Solve $P(x_1^{'}, x_2^{'} \ldots x_{i-1}^{'})$ using the parametric approach. Let $(x_2^{'} \ldots x_n^{'})$ be its optimal solution and B_i be its optimal basis.
v. If $x_i^{'} = x_i$,Stop. $(x_1^{'}, x_2^{'} \ldots x_n^{'})$ is a global optimum and is the current best point of (IR).

Step 1.

i. Given the basis B_i solve $P_R(B_i)$ using the two-phase method.
ii. If $P_R(B_i)$ is feasible, and any of its optimal solutions verify constraints (2) and (3), then go to step 2.
iii. If $P_R(B_i)$ is feasible, and none of its optimal solutions verify constraints (2) and (3), then go to step 3.
iv. If $P_R(B_i)$ is not feasible, then go to step 4.

Step2

i. Compare this optimal solution with the current best point of (IR) and update, if necessary, the latter.
ii. Compute C_1.
iii. If C_1=0. Stop. The current best point of (IR) is a global optimum to MLLFP.
iv. Set $E_1 = E_1 \cup \{C_1\}$, then go to step 5.

Step 3.

Compute C_3. Set $E_3 = E_3 \cup \{C_3\}$, then go to step 5.

Step 4.

Compute C_2. Set E_2= $E_2 \cup \{C_2\}$.

Step 5.

i. Solve P2.
ii. If P2 is not feasible, stop. The current best point of (IR) is a global optimum to MLLFP.
iii. Let D be the constructed set of vectors. If rank (D) = m, then go to step 6, otherwise go to step 7.

Step 6.

i. Set B_i=D. Compute $\left[\lambda_i^1, \lambda_i^u\right]$ by checking condition (1).
ii. If condition (1) is not verified, compute C_3. Set $E_3 = E_3 \cup \{C_3\}$ and then go to step 5. Otherwise go to step 1.

Step 7.

i. Let rank (D)=k. Let D' be the matrix of independent vectors of D. Set D=D'. Check the existence of a set G of (m-k) vectors of A^i so that B_i=[D,G] is a basis from A^i verifying conditions given by sets E_1,E_2 and E_3 and condition (1) for $\lambda_i \in \left[\lambda_i^1, \lambda_i^u\right]$.
ii. If it exists, then go to step 1. Otherwise compute C_3, Set $E_3 = E_3 \cup \{C_3\}$ and Go to step 5.

Step 8.

After getting an optimal solution for a particular level go to step 0.3 and repeat this procedure until i=n. At last, we get the optimal solution for the given problem.

5. Numerical Example

For illustration of the proposed methodology, we consider the following example:

$$\underset{(x_1,x_2)}{Min}.f_1 = -6x_1 - 2x_2 + 2x_3 - 20x_4 - 2x_5 + 8x_6 - 6x_7$$

Where

$$\underset{(x_3,x_4)}{Min}.f_2 = \frac{(3 + 2x_1 + 2x_2 + 4x_3 - 2x_4 + 2x_5 - 2x_6 + 6x_7)}{(8 + 4x_1 + 2x_3 + 2x_4 - 6x_5 + 4x_6 - 8x_7)}$$

$$\underset{(x_5,\ldots x_{10})}{Min}.f_2 = \frac{(1 + x_1 + x_2 + 3x_3 - 5x_4 + 4x_5 - x_6 + 5x_7)}{(5 + 2x_1 + 2x_2 + 3x_4 - 2x_5 + 2x_6 - 7x_7)}$$

s.t.

$$-x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 1$$
$$2x_1 - x_3 + 2x_4 - x_5 + x_6 + 2x_7 + x_9 = 1$$
$$2x_2 + 2x_3 - x_4 - x_5 - x_6 - 2x_7 + x_{10} = 1$$

$x_i \geq 0,$ i=1,2,3…10.

Solution

The optimal solution to problem P_R is = (0,0,2,2,1,0,0,0,0,0). The optimal value of the objective function is -38, which constitutes a lower bound on the objective function of P1.By fixing $x_1^{'}$ = (0,0) and solving the fractional problem of the second level we get the optimal solution $x_2^{'}$ = (0,0,0,1,0,0,0,2) Since $x_2^{'} \neq x_2$, next we go to step 1.

The current best point of (IR) is $f_1 =$ 8, Basis B_1 is given by vectors with indices 4,6,10 will be the first analyzed basis $\left[\lambda_1^1, \lambda_1^u\right]$ = [-4/5, 1/3].

After the first iteration we get the optimal solution (0,1,0,0,0,1,0,0,0,0).

All the results are shown in the following tables:

Table 1: Optimal tableaus of Problems P_R (B_1^i)

	Basic Variables	Value	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	a_1	a_2
Iteration 1.	x_6	1	-2	0	-1	0	3	1	0	2	-1	0		
	x_4	0	2	0	0	1	-2	0	1	-1	1	0		
	x_2	1	0	1	1/2	0	0	0	-1/2	1/2	0	1/2		
			5	0	11	0	-66	0	13	-35	28	1		
(0,1,0,0,0,1,0,0,0,0) $\in$ (IR) f_1=6.														
Iteration 2.	x_5	1	0	0	-1	1	1	1	1	1	0	0		
	x_1	1	1	0	1	3/2	0	1	3/2	1/2	1/2	0		
	x_2	1	0	1	1/2	0	0	0	-1/2	1/2	0	1/2		
			0	0	-5	-9	0	16	4	6	3	1		
(1,1,0,0,1,0,0,0,0,0) $\in$ (IR) f_1= -10														
Iteration 3.	P_R (B_1^3) is not feasible.													
a_1		½	-1	0	-1/2	0	3/2	½	0	1	-1/2	0	1	
	x_7	1/2	1	0	-1/2	1	-1/2	1/2	1	0	1/2	0	0	
	x_2	1	0	1	1/2	1/2	-1	0	0	0	1/2	1/2	0	

Table 2: Summary of the algorithm

	C_1	C_2	C_3	D is formed by vectors with indices	Interval of λ
Iter. 1	{5,8}			{5,6,8}	$\left[-1/3, \infty\right]$

Iter. 2	{3,4}	{3,4,7}	$[-3/4, \infty]$

Table 3: Final table showing infeasibility of third level

Basic Variables	Value	x_5	x_6	x_7	x_8	x_9	x_{10}	a_1
x_8	0	0	0	-1	1	0	1	0
a_1	0	0	-2	-4	0	-1	1	1
x_5	1	1	1	2	0	0	-1	0
	0	2	4	0	1	0	0	

CONCLUSION

In this study, we proposed a basis examination method to find the global optima of the multi-level linear/ linear fractional programming problem (MLLFP) in which the objective function of the first level is linear and the objective functions of other levels are linear fractional and the feasible region is a polyhedron. We conclude that:

- Under the usual assumptions there is an extreme point of the polyhedron S which solves the given problem.
- One of its most outstanding features is that this implicit search is only made among basis of the technological coefficient submatrix corresponding to variables of each level one by one.
- The inducible region of MLLFP is formed by the union of connected faces of the feasible region and optimal solution to MLLFP occurs at an extreme point of polyhedron S.

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