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# INITIAL VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATION BY USING ADOMIAN DECOMPOSITION METHOD

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## Abstract

*Solution of initial value problems of fractional differential equations with Adomian decomposition method is an emerging area of present day research as these equations are being used in various applied fields. The Adomian Decomposition Method is a semi-analytical method for solving ordinary and partial non-linear differential equations. The aim of this method is towards unified theory for the solution of partial differential equation. The aim which has been superseded by the more general theory of the homotopy analysis method. In this paper, we worked on Adomian decomposition method to solve the initial value problems of linear and non-linear fractional differential equations.*

## Keywords

*Caputo fractional derivative, Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Adomian decomposition method, initial value problems.*

## Introduction

The Adomian decomposition method has been used to solve various scientific models. The Adomian decomposition method yields rapidly convergent series solution with much less computational work. Unlike the traditional numerical methods, the Adomian decomposition method needs no discretisation, linearization, transformation or perturbation. Recently, more attention devoted to the search for reliable and more efficient solution methods for equations modelling physical phenomena in various fields of science and engineering [6,7]. One of the method which has been received much concern is the Adomian decomposition method.

In this work, our emphasis is todetermine the accuracy and efficiency of the Adomian decomposition method in solving initial value problems of linear and non-linear fractional differential equation.

# Preliminary Concept:

## Riemann-Liouville fractional integral

### Definition

Suppose that $f(x) \in C([a,b])$, $a < x < b$ then the Riemann-Liouville fractional integral of order $\alpha$ of $a$ function $f(x)$ is defined by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \cdots\cdots (1)$$

where $\alpha \in ]-\infty, \infty[$

This formula represents the integral of arbitrary order $\alpha > 0$, but does not permit order $\alpha = 0$ because it formally corresponds to the identity operator.

## Riemann-Liouville Fractional Derivative

Recently many models are formulated in terms of fractional derivatives, such as in control processing, viscoelasticity, signal processing and anomalous diffusion.

### Definition

The Riemann-Liouville fractional derivative [4] of a function $f(x)$, where $f(x) \in C([a,b]) and \ a < x < b$ with fractional order $\alpha, \alpha \in ]0,1[$ is defined as

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} \, dt \cdots\cdots (2)$$

This is called the Riemann-Liouville fractional derivative of arbitrary order $\alpha$.
If $0 < \alpha < 1$ i.e., $\alpha \in ]0,1[$
then $D_{a+}^\alpha f(x)$ exists for all $f \in C'([a,b])$ all $x \in ]a,b[$

### Lemma 1:

for $\alpha, \beta \geq 0$, $f(x) \in L_1[0,T]$ then
$I_{0^+}^\alpha I_{0^+}^\beta f(x) = I_{0^+}^{\alpha+\beta} f(x) = I_{0^+}^\beta I_{0^+}^\alpha f(x)$
is satisfied almost everywhere on $[0,T]$. Moreover, if $f(x) \in L_1[0,T]$ then the above equation is true for all $x \in [0,T]$.

# Caputo Fractional Derivative

## Definition

Mathematically [1,2,3] it is defined as,
Suppose that, $\alpha > 0,\ x > a,\ \alpha, a, x\ \in\ \mathbb{R}$

$$D_*^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\, dt, & n-1 < \alpha < n \in N \\ \frac{d^n f(x)}{dx^n} & \alpha = n\ \in N \\ I_{a^+}^{\alpha}\ f(x) & \alpha \leq 0 \end{cases} \qquad \cdots\cdots\cdots\cdots (3)$$

is called the Caputo fractional derivative or Caputo fractional differential operator of $\alpha$.

**Lemma 2:**

$$\text{If}\ \ \alpha > 0,\ f(x) \in L_1[0,T].$$
$$\text{then,}\ \ {}^{c}D_{0+}^{\alpha}\ I_0^{\alpha}\ f(x) = f(x)\ \ \text{for all}\ \ x \in [0,T]$$

**Proof:**
By the definition of Riemann-Liouville fractional derivative, by using equation (2),

$$D_{0^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}}\, dt$$

by substituting $\alpha = n, f(x) = x \Rightarrow f(t) = t$
$(x-t) = u$
$\Rightarrow -dt = du$
Also, new limit point will be
when $t = x \Rightarrow u = 0$
when $t = 0 \Rightarrow u = x$

$\therefore$ The above equation becomes,

$$\begin{aligned}
D_*^\alpha =& \frac{1}{\Gamma(1-n)} \ \frac{d}{dx} \ \int_x^0 \ \frac{(x-u)}{(u)^n} \ (-du) \\
=& \frac{1}{\Gamma(1-n)} \ \frac{d}{dx} \ \int_0^x \ \frac{(x-u)}{(u)^n} \ (du) \\
=& \frac{1}{\Gamma(1-n)} \ \frac{d}{dx} \ \int_0^x \ (x-u) \ u^{-n} \ du
\end{aligned}$$

Now, using integration by parts,

$$\begin{aligned}
D_*^\alpha f(x) =& \frac{1}{\Gamma(1-n)} \frac{d}{dx} \int_0^x \left[ (x-u) \int_0^x \ u^{-n} du - \left( \int_0^x \frac{d(x-u)}{du} \int_0^x \ u^{-n} du \right) du \right] \\
=& \frac{1}{\Gamma(1-n)} \ \frac{d}{dx} \ \int_0^x \ \frac{u^{(1-n)}}{(-n+1} \ (du) \\
=& \frac{1}{(-n+1)\Gamma(1-n)} \ \frac{d}{dx} \ \left[ \frac{u^{-n+1+1}}{-n+1+1} \right]_0^x \\
=& \frac{1}{(-n+1)\Gamma(1-n)} \ \frac{(-n+2)x^{-n+1}}{(-n+2)}
\end{aligned}$$

Hence,

$$D_*^\alpha f(x) = \frac{x^{-n+1}}{(-n+1)\Gamma(1-n)} \cdots\cdots (4)$$

Secondly, by the definition of Riemann-Liouville fractional integral of a function $f(x)$

$$I_{0^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$$

Let us put $\alpha = n, f(x) = x \Rightarrow f(t) = t$
$(x-t) = u$
$\Rightarrow -dt = du$
when $t = x \Rightarrow u = 0$

when $t = 0 \Rightarrow u = x$
$\therefore$ The above integral becomes,

$$
\begin{aligned}
I^{n}_{0^+}(x) =& \frac{1}{\Gamma(n)} \int_0^x \frac{x-u}{u^{1-n}} du \\
=& \frac{1}{\Gamma(n)} \int_0^x (x-u)\ u^{n-1} du \\
=& \frac{1}{\Gamma(n)} \left[ \frac{(x-u)u^n}{n} \right]_0^x \ - \ \int_0^x \left( \frac{d(x-u)}{du}\ \frac{u^n}{n} \right) du \\
=& \frac{1}{n(n+1)\Gamma(n)}\ x^{n+1} \\
I^{n}_{0^+}(x) =& \frac{1}{(n+1)\Gamma(n+1)}\ x^{n+1} \qquad \cdots\cdots (5)
\end{aligned}
$$

Now,
Multiplying the Riemann-Liouville derivative operator to the equation (5) and using (4), we get

$$
\begin{aligned}
D^{n}_{0^+} I^{n}_{0^+}(x) =& D^{n}_{0^+} \left[ \frac{x^{n+1}}{(n+1)\Gamma(n+1)} \right] \\
=& D^{n}_{0^+} \left[ \frac{x^n x}{(n+1)\Gamma(n+1)} \right] \\
=& \frac{x^n}{(n+1)\Gamma(n+1)}\ (D^{n}_{0^+} x) \\
=& \frac{x^n}{(n+1)\Gamma(n+1)}\ \frac{x^{-n+1}}{(1-n)\Gamma(1-n)}
\end{aligned}
$$

Hence, $\quad D^{n}_{0^+} I^{n}_{0^+}(x) = f(x)$

## Adomian Decomposition Method

Consider the differential equation

$$Lv + Rv + Nv = g \cdots\cdots (6)$$

where,
L - highest order derivative & easily invertible.
R - linear differential operator of order less than L.
Nv - represents the non-linear terms.
g - source term.
The functions $v(x)$ is supposed to be bounded for all $x \in I = [0, T]$ & the nonlinear term Nv satisfies Lipschitz condition i.e.,
for initial value problem, we conveniently define $L^{-1}$ for $L = \frac{d^n}{dt^n}$ as the $n$-fold definite integral from zero to $x$. If L is second order operator, $L^{-1}$ is a two fold integral & so by solving for v.
$|Nv - Nw| \leq k_1 |v - w|$
where $k_1$ is a positive constant.
Since L is invertible there, we get,

$$L^{-1}Lv = L^{-1}g - L^{-1}Rv - L^{-1}Nv + B \cdots\cdots\cdots (7)$$

where,
$B$ is the constant of integration & satisfies $LB = 0$ .

$$L^{-1}(\cdot) = \int_0^x (\cdot)\ dx$$

Now, the Adomian decomposition method consists of approximating the solution of (6) as an infinite series

$$v(y, x) = \sum_{n=0}^{\infty} v_n(y, x) \cdots\cdots\cdots (8)$$

and the nonlinear term Nv will be decomposed by the infinite series of Adomian polynomials.

$$Nv = \sum_{n=0}^{\infty} An(v_0, v_1, v_2, \cdots, v_n) \cdots\cdots\cdots (9)$$

where $An$ is Adomian polynomial calculated by using the formula,

$$An = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\ N(w(\lambda))\right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots$$

where,

$$w(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n$$

Substituting the decomposition series i.e. equation (8) & (9) in (7).
We get,

$$\sum_{n=0}^{\infty} v_n(y,x) = B + L^{-1}g - L^{-1}R\left(\sum_{n=0}^{\infty} v_n(y,x)\right) - \sum_{n=0}^{\infty} A_n(v_0, v_1 \cdots v_n) \cdots (10)$$

from the above equation, we can say that,

$$\begin{aligned} v_0 =& B + L^{-1}g \\ v_1 =& -L^{-1}(Rv_0) - L^{-1}(A_0) \\ v_2 =& -L^{-1}(Rv_1) - L^{-1}(A_1) \\ &. \\ &. \\ &. \\ v_{n+1} =& L^{-1}(Rv_n) - L^{-1}(A_n), \quad n \geq 0 \end{aligned}$$

where $B$ is the initial condition.
Hence all terms of V are can be found & the general solution obtained by using Adomian decomposition method as

$$v(y,x) = \sum_{n=0}^{\infty} v_n(y,x) \cdots\cdots (11)$$

The convergence of the series [5]has been proved.
Now,we apply Adomian decomposition method to derive the solution of fractional partial differential equations. We solve few examples by Adomian decomposition method.
Firstly, we apply the Adomian decomposition method to obtain approximate solution of initial value problems for fractional BBM-Burgers equation with $\epsilon = 1$.

**Example 1:**
Consider the following nonlinear fractional differential equation:

$$\frac{\partial^\alpha v}{\partial x^\alpha} - \frac{\partial^3 v}{\partial y^3} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$(y,x) \in w \times [0,T]$ & $0 < \alpha \leq 1$
with initial condition,
$v(y,0) = f(y)$ where $f(y) = \cos y$

**Solution**
Given nonlinear fractional differential equation is,

$$\frac{\partial^\alpha v}{\partial x^\alpha} - \frac{\partial^3 v}{\partial y^3} + \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial^\alpha v}{\partial x^\alpha} = - v\frac{\partial v}{\partial y} + \frac{\partial^3 v}{\partial y^3} - \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y}$$

$$= - v(y,x)L_y v(y,x) + L_{yyy}v(y,x) - L_{yy}v(y,x) + L_y v(y,x)$$

where,

$$L_{yyy} = \frac{\partial^3}{\partial y^3},\ L_{yy} = \frac{\partial^2}{\partial y^2},\ L_y = \frac{\partial}{\partial y}$$

by the definition of Caputo fractional derivative $D_x^\alpha$ & we know that $I^\alpha$ is inverse of the operator $D_x^\alpha$.
Now, applying $I^\alpha$ to the both sides of the given equation,
we get,

$$v(y,x) = I^\alpha(Nv) + I^\alpha(L_{yyy}v) - I^\alpha(L_{yy}v) + I^\alpha(L_y v) + B$$

where

$$Nv = v\frac{\partial v}{\partial y}$$

The first few terms of the Adomian polynomials are given by:

$$
\begin{aligned}
A_0 =& v_0 \frac{\partial v_0}{\partial y}\\
A_1 =& v_0 \frac{\partial v_1}{\partial y} \ + v_1 \frac{\partial v_0}{\partial y}\\
A_2 =& v_0 \frac{\partial v_2}{\partial y} \ + v_1 \frac{\partial v}{\partial y} \ + v_2 \frac{\partial v_0}{\partial y}\\
&.\\
&.\\
&.
\end{aligned}
$$

and so on.

using equation (10), we get,

$$
\begin{aligned}
\sum_{n=0}^{\infty} \ v_n(y,x) =& B - I^{\alpha}\Big(\sum_{n=0}^{\infty} A_n(v_0, v_1 \cdots v_n)\Big) \ + I^{\alpha}\Big(\sum_{n=0}^{\infty} L_{yyy} v_n\Big)\\
&-I^{\alpha}\Big(\sum_{n=0}^{\infty} L_{yy} v_n\Big) \ + I^{\alpha}\Big(\sum_{n=0}^{\infty} L_{y} v_n\Big)
\end{aligned}
$$

It is clear that,

$$
\begin{aligned}
v_1 =& - I^{\alpha} A_0 \ + I^{\alpha} L_{yyy} v_0 \ - I^{\alpha} L_{yy} v_0 \ + I^{\alpha} L_y v_0\\
v_2 =& - I^{\alpha} A_1 \ + I^{\alpha} L_{yyy} v_1 \ - I^{\alpha} L_{yy} v_1 \ + I^{\alpha} L_y\\
&.\\
&.\\
&.\\
v_{n+1} =& - I^{\alpha} A_n \ + I^{\alpha} L_{yyy} v_n \ - I^{\alpha} L_{yy} \ + I^{\alpha} L_y v_n
\end{aligned}
$$

by putting the value of $v_0, v_1, v_2 \cdots$ , from above, we get the solution of initial value problem

$$
\begin{aligned}
v(y,x) =& v_0 + v_1 + V_2 + v_3 + \cdots + v_n + \cdots \\
v_0 =& v(y,0) = f(y) = \cos y \\
v_1 =& - I^\alpha A_0 \; + I^\alpha (L_{yyy} v_0) \; - I^\alpha (L_{yy} v_0) \; + I^\alpha (L_y v_0) \\
v_1 =& \frac{f(y) + f'(y) - f'''(y) + f''(y) - f'(y)}{\Gamma(\alpha+1)} \; x^\alpha \\
\therefore v_1 =& f_1(y) \; \frac{x^\alpha}{\Gamma(\alpha+1)}
\end{aligned}
$$

where,

$$
\begin{aligned}
f_1(y) =& f(y) + f'(y) - f'''(y) + f''(y) - f'(y) \\
=& (-\sin y - 1)\cos y - 2\sin y \\
v_2 =& - I^\alpha (A_1) \; + I^\alpha (L_{yyy} v_1) \; - I^\alpha (L_{yy} v_1) \; + I^\alpha (L_y v_1) \\
=& \frac{f_2(y)}{\Gamma(2\alpha+1)} \; x^{2\alpha}
\end{aligned}
$$

where,

$$
\begin{aligned}
f_2(y) =& - \left[f(y) + f'{}_1(y) + f_1(y)f'(y) - f_1'''(y) + f_1''(y) - f_1'(y)\right] \\
=& \cos^3 y + 3cos^2 y + \big((-\sin y - 1)\sin y - 2\sin y - 1\big)\cos y \\
& - 5\sin^2 y - 2\sin y,
\end{aligned}
$$

*Similarly,*

$$
\begin{aligned}
v_3 =& f_3(y)\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} \\
&. \\
&. \\
&. \\
v_n =& f_n(y)\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}
\end{aligned}
$$

Summing all these terms we get the solution of the equation.

$$
\begin{aligned}
v(y,x) =& f(y) + \frac{x^{\alpha}}{\Gamma(\alpha+1)}\ f_1(y) + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \\
& \cdots \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} f_n(y) + \cdots \\
v(y,x) =& \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} f_n(y)
\end{aligned}
$$

where $f_0(y)$ is an initial condition.

**Example 2:**
Consider the system of initial value problem of fractional equations.

$$D_x^\alpha v = vD_y v + wD_z v$$

$$D_x^\alpha w = vD_y w + wD_z w$$

where $0 < \alpha \leq 1$ & $(y, x) \in \Omega(0, T]$
& with initial condition

$$v(y, z, 0) = f(y, z)$$

$$w(y, z, 0) = g(y, z), \quad y, z \in \Omega$$

Note that, $\Omega = (0, 1)$
**Solution**
The above system can be written in the equivalent form

$$\frac{\partial^\alpha v}{\partial x^\alpha} = v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial Z}$$
$$\frac{\partial^\alpha w}{\partial x^\alpha} = v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial Z}$$

$$\therefore N_1(v, w) = v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial Z}$$
$$\& \ N_2(v, w) = v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial Z}$$

$$\therefore Lv = N_1(v, w)$$

$$\& \ Lw = N_2(v, w)$$

Applying $L^{-1}(\cdot) = I^\alpha$ to the both sides of above,
we get,

$$v(y, z, x) = \phi + I^\alpha N_1(v, w)$$

$$\& \quad w(y, z, x) = \phi + I^\alpha N_2(v, w)$$

where the nonlinear operator $N_1(v,w)$ & $N_2(v,w)$ can be written in the decomposition form

$$N_1(v,w) = \sum_{n=0}^{\infty} A_n(v_0, v_1, \cdots v_n)$$
$$N_2(v,w) = \sum_{n=0}^{\infty} B_n(v_0, v_1, \cdots v_n)$$

where $A_n$ & $B_n$ are the Adomian polynomials which has the following form

$$\begin{aligned} A_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\ N_1(v,w)\left(\sum_{i=0}^{\infty}\ \lambda^i v_i\right)\right]_{\lambda=0} \\ B_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\ N_2(v,w)\left(\sum_{i=0}^{\infty}\ \lambda^i v_i\right)\right]_{\lambda=0} \qquad , n = 0, 1, 2, \cdots \end{aligned}$$

Generalizing these Adomian polynomials,
we get

$$\begin{aligned} A_n =& \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\left(\frac{\partial}{\partial y}\sum_{i=0}^{\infty}\lambda^i v_i\right)+\right. \\ & \left.\left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\left(\frac{\partial}{\partial y}\sum_{i=0}^{\infty}\lambda^i v_i\right)\right]_{\lambda=0} \quad , \ n = 0, 1, 2, 3, \cdots \end{aligned}$$

we have,

$$\begin{aligned} A_0 &= v_0\ \frac{\partial v_0}{\partial y}\ +\ w_0\ \frac{\partial v_0}{\partial z} \\ A_1 &= v_0\ \frac{\partial v_1}{\partial y}\ +\ w_0\ \frac{\partial v_1}{\partial z}\ + v_1\frac{\partial v_0}{\partial y} + w_1\frac{\partial v_0}{\partial z} \\ A_2 &= v_0\ \frac{\partial v_2}{\partial y}\ +\ w_0\ \frac{\partial v_2}{\partial z}\ + v_1\frac{\partial v_1}{\partial y} + w_1\frac{\partial v_1}{\partial z} + \ \frac{\partial v_2}{\partial y}\ + w_2\frac{\partial v_0}{\partial z} \end{aligned}$$

Similarly,

$$A_3 = v_0 \frac{\partial v_3}{\partial y} + w_0 \frac{\partial v_3}{\partial z} + v_1 \frac{\partial v_2}{\partial y} + w_1 \frac{\partial v_2}{\partial z} + v_2 \frac{\partial v_1}{\partial y} + w_2 \frac{\partial v_1}{\partial z} + v_3 \frac{\partial v_0}{\partial y} + w_3 \frac{\partial v_0}{\partial z}$$

.

.

.

and so on,

Now,
To solve this problem again we have to generalize these Adomian polynomials as follows,

$$B_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\sum_{i=0}^{\infty}\lambda^i v_i\right)\left(\frac{\partial}{\partial x}\sum_{i=0}^{\infty}\lambda^i w_i\right) + \left(\sum_{i=0}^{\infty}\lambda^i w_i\right)\left(\frac{\partial}{\partial z}\sum_{i=0}^{\infty}\lambda^i w_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, 3, \cdots$$

$$\therefore B_0 = v_0 \frac{\partial v_0}{\partial y} + w_0 \frac{\partial w_0}{\partial z}$$

$$B_1 = v_0 \frac{\partial w_1}{\partial y} + w_0 \frac{\partial w_1}{\partial z} + V_1 \frac{\partial w_0}{\partial y} + w_1 \frac{\partial w_0}{\partial z}$$

$$B_2 = v_0 \frac{\partial w_2}{\partial y} + w_0 \frac{\partial w_2}{\partial z} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} + v_2 \frac{\partial w_0}{\partial y} + v_2 \frac{\partial w_0}{\partial z}$$

$$B_3 = v_0 \frac{\partial w_3}{\partial y} + w_0 \frac{\partial w_3}{\partial z} + v_1 \frac{\partial w_2}{\partial y} + w_1 \frac{\partial w_2}{\partial z} + v_2 \frac{\partial w_1}{\partial y} + w_2 \frac{\partial w_1}{\partial z} + v_3 \frac{\partial w_0}{\partial y} + w_3 \frac{\partial w_0}{\partial y}$$

.

.

.

and so on,

from equation (10) of Adomian decomposition method,
we obtain,

$$\sum_{n=0}^{\infty} v_n(y,z,x) = v(y,z,0) + I^{\alpha}\Big(\sum_{n=0}^{\infty} A_n(v_0, v_1, \cdots v_n)\Big) \&$$
$$\sum_{n=0}^{\infty} w_n(y,z,x) = w(y,z,0) + I^{\alpha}\Big(\sum_{n=0}^{\infty} B_n(w_0, w_1, \cdots w_n)\Big)$$

The associated decomposition is given by,

$$v_0 = v(y,z,0),\ \ v_{n+1} = I^{\alpha}\big(N_1(V_n, w_n)\big)$$
$$w_0 = w(y,z,0),\ \ w_{n+1} = I^{\alpha}\big(N_2(v_n, w_n)\big), n = 0, 1, 2, \cdots$$

Then using above equations we get,

$$v_0 = v(y,z,0)$$
$$v_1 = I^{\alpha} A_0$$
$$v_2 = I^{\alpha} A_1$$

.

.

.

$$v_{n+1} = I^{\alpha} A_n$$

Similarly,

$$
\begin{aligned}
w_0 &= w(y,z,0)\\
w_1 &= I^{\alpha}B_0\\
w_2 &= I^{\alpha}B_1\\
&\cdot\\
&\cdot\\
&\cdot\\
w_{n+1} &= I^{\alpha}B_n
\end{aligned}
$$

using these values of $v_0, v_1, v_2 \cdots v_n$ & $w_0, w_1, \cdots w_n$ we can find a solution of given initial value problem

$$
\begin{aligned}
v_0 =& v(y,z,0) = f(y,z)\\
v_1 =& I^{\alpha}(A_0)\\
=& \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-t)^{1-\alpha}} \left[ v_0 \frac{\partial v_0}{\partial y} + w_0 \frac{\partial v_0}{\partial z} \right] dt\\
=& f_1(y,z) \frac{x^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}
$$

where,

$$
f_1(y,z) = -\left[ f(y,z) \frac{\partial f(y,z)}{\partial y} + g(y,z) \frac{\partial f(y,z)}{\partial z} \right]
$$

$$
\begin{aligned}
v_2 =& I^{\alpha}(A_1)\\
=& f_2(y,z) \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}
\end{aligned}
$$

where,

$$f_2(y,z) = \Bigg[f(y,z)\ \frac{\partial f_1(y,z)}{\partial y}\ + g(y,z)\frac{\partial f_1(y,z)}{\partial z} +$$
$$f_1(y,z)\ \frac{\partial f(y,z)}{\partial y} + g(y,z)\frac{\partial f(y,z)}{\partial z}\Bigg]$$

Similarly, we can find $w_0, w_1, w_2, \cdots$

$$\begin{aligned}
w_0 =& w(y,z,0) = g(y,z)\\
w_1 =& I^{\alpha} B_0\\
=& \frac{1}{\Gamma(\alpha)}\ \int_0^x\ \frac{1}{(x-t)^{1-\alpha}}\ \left[v_0\frac{\partial w_0}{\partial y}\ + w_0\frac{\partial w_0}{\partial z}\right]\ dt\\
=& g_1(y,z)\frac{x^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}$$

*where,*

$$g_1(y,z) = -\left[f(x,y)\ \frac{\partial f(y,z)}{\partial y}\ + g(y,z)\frac{\partial g(y,z)}{\partial z}\right]$$

$$\begin{aligned}
w_2 =& I^{\alpha}(B_1)\\
=& g_2(y,z)\ \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}
\end{aligned}$$

*where,*

$$g_2(y,z) = f(y,z)\ \frac{\partial g_2(y,z)}{\partial y}\ + g(y,z)\frac{\partial g_1(y,z)}{\partial z} +$$
$$f_1(y,z)\ \frac{\partial g(y,z)}{\partial y} + g_1(y,z)\frac{\partial g(y,z)}{\partial z}$$

Summing all these terms we get,

$$\begin{aligned} v(y,z,x) =& \sum_{n=0}^{\infty} \ v_n = v_0 + v_1 + v_2 + \cdots + v_n + \cdots \\ =& f(y,z) + f_1(y,z)\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \ f_2(y,z)\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \\ & f_3(y,z)\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \\ \therefore v(y,z,x) =& \sum_{n=0}^{\infty} f_n(y,z)\frac{x^{n\alpha}}{\Gamma(n\alpha+1)} \end{aligned}$$

Similarly,

$$\begin{aligned} w(y,z,x) =& \sum_{n=0}^{\infty} \ w_n = w_0 + w_1 + w_2 + w_3 + \cdots + w_n + \cdots \\ =& g(y,z) + g_1(y,z)\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \ g_2(y,z)\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \\ & g_3(y,z)\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots + g_n(y,z)\frac{x^{n\alpha}}{\Gamma(n\alpha+1)} \\ \therefore w(y,z,x) =& \sum_{n=0}^{\infty} g_n(y,z)\frac{x^{n\alpha}}{\Gamma(n\alpha+1)} \end{aligned}$$

This gives the solution of the initial value problem.

Finally, we apply the Adomian decompostion method to obtain approximate solution of initial value problems for fractional BBM-Burgers equation with $\epsilon = 1$

**Example 3:**

Consider the initial value problem for fractional BBM-Burger's equation of the form [5-11]

$$\frac{\partial^{\alpha} v}{\partial x^{\alpha}} \ - \frac{\partial^2 v}{\partial y^2} + v\frac{\partial v}{\partial y} = 0$$

where,
$0 < \alpha \leq 1$ & with initial condition,
$v(y,0) = \phi = f(y) = \sin(y), y \in \Omega \times (0,T]$
Note that here,

$$\Omega = (0,1)$$

**Solution:**

Given fractional BBM-Burger's equation is

$$\frac{\partial^\alpha v}{\partial x^\alpha} - \frac{\partial^2 v}{\partial y^2} + v\frac{\partial v}{\partial y} = 0$$

In an operator form it can be written as,

$$\begin{aligned}\frac{\partial^\alpha v}{\partial x^\alpha} &= \frac{\partial^2 v}{\partial y^2} - v\frac{\partial v}{\partial y}\\ \frac{\partial^\alpha v}{\partial x^\alpha}(y,z) &= \big(L_{yy}v(y,x)\big) - \big(v(y,x)L_y v(y,x)\big)\end{aligned}$$

where

$$L_{yy} = \frac{\partial^2}{\partial y^2},\ L_y = \frac{\partial}{\partial y}$$

& the fractional differential operator $\frac{\partial^\alpha}{\partial x^\alpha}$ is defined in the definition of Caputo fractional differential operator. We know that $I^\alpha$ is the inverse of the operator $\frac{\partial^\alpha}{\partial x^\alpha}$.
Now, applying $I^\alpha$ to the both sides of our equation,
we obtain,

$$v(y,x) = I^\alpha(L_{yy}v) - I^\alpha(Nv) + \phi$$

$$\text{where } Nv = v\frac{\partial v}{\partial y}$$

In order to solve our problem we must generalize these Adomian polynomials as follows,

$$\begin{aligned}
A_n =& \frac{1}{n!}\ \frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{n}\ \lambda^i v_i\right)\frac{\partial}{\partial y}\left(\sum_{i=0}^{n}\ \lambda^i v_i\right)\right]_{\lambda=0} \quad , n = 0, 1, 2, \cdots \\
\therefore A_0 =& v_0\frac{\partial v_0}{\partial y} \\
A_1 =& v_0\frac{\partial v_1}{\partial y} + v_1\frac{\partial v_0}{\partial y} \\
A_2 =& v_0\frac{\partial v_2}{\partial y_1} + v_1\frac{\partial v_1}{\partial y} + v_2\frac{\partial v_0}{\partial y} \\
& . \\
& . \\
& . \\
& \text{and so on,}
\end{aligned}$$

Thus,

$$\begin{aligned}
v(y,x) =& \phi + I^\alpha\left(\sum_{i=0}^{n}(L_{yy}v_n)\right) - I^\alpha\left(\sum_{i=0}^{n}(A_n)\right) \\
v_0 =& \phi = v(y,0) \\
v_1 =& I^\alpha(L_{yy}v_0) - I^\alpha(A_0) \\
v_2 =& I^\alpha(L_{yy}v_1) - I^\alpha(A_1)
\end{aligned}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$v_{n+1} = I^{\alpha}(L_{yy}v_n) - I^{\alpha}(A_n)$$

and so on,

Consequently,

$$v(y,x) = v_0 + v_1 + v_2 + v_3 + \cdots v_n + \cdots$$

Finding $v_0, v_1, v_2, \cdots$ by using given initial conditions, we get

$$\begin{aligned} v_0 =& v(y,0) = f(y) = \sin y \\ v_1 =& I^{\alpha}(L_{yy}v_0) - I^{\alpha}(A_n) \\ =& f_1(y)\frac{x^{\alpha}}{\Gamma(\alpha+1)} \end{aligned}$$

where,

$$\begin{aligned} f_1(y) &= -f''(y) + f(y)f'(y) \\ &= \sin y + \sin y \cos y \\ &= \sin y(1 + cosy) \end{aligned}$$

also,

$$\begin{aligned} v_2 =& I^{\alpha}(L_{yy}v_1) - I^{\alpha}(A_1) \\ =& f_2(y)\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \end{aligned}$$

where,

$$
\begin{aligned}
f_2(y) &= -f_1''(y) - f(y) - f_1'(y)f'(y) \\
&= \sin y + \big[ -1 - 5\cos y - \cos^2 y - (1 + cosy)\cos y \big]
\end{aligned}
$$

$$
\begin{aligned}
\& \quad v3 =& I^\alpha(L_{yy}v_2) - I^\alpha(A_2) \\
=& f_3(y)\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}
\end{aligned}
$$

$Similarly,$

$$
\begin{aligned}
v_4 =& f_4(y)\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} \\
& . \\
& . \\
& . \\
v_n =& f_n(y)\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}
\end{aligned}
$$

by summing all these terms we will have the solution of the given initial value problem is

$$
\begin{aligned}
v(y,x) =& f(y) + \frac{x^\alpha}{\Gamma(\alpha+1)}f_1(y) + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\ f_2(y) + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)}f_3(y) \\
& + \cdots + \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}f_n(y)\ + \cdots \\
\therefore v(y,x) =& \sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}f_n(y)
\end{aligned}
$$

This completes the solution of BBM-Burger equation.

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