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APPLICATION OF ADOMIAN DECOMPOSITION METHOD FOR SOLVING INITIAL VALUE PROBLEMS IN FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

The Adomian Decomposition Method is a semi-analytical method for solving ordinary and partial non-linear differential equations. The crucial aspect of the method is employment of the 'Adomian polynomials', which allow for solution convergence of the nonlinear portion of the equation without simply linearizing the system. These polynomials mathematically generalise to Maclaurin series about an arbitrary external parameter which gives the solution method more flexibility than direct Taylor series expansion.In the present paper, we foundthe solution of nonlinear initial value problems in series form by using Adomian decomposition method.

Keywords

Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative, Adomian decomposition method, initial value problems.

Introduction

Recently, more attention devoted to the search for reliable and more efficient solution methods for equations modelling physical phenomena in various fields of science and engineering [6,7].One of the methods which has received much concern is the Adomian decomposition method.The Adomian decomposition method has been used to solve various scientific models. The Adomian decomposition method yields rapidly convergent series solution with much less computational week. Unlike the traditional numerical methods, the Adomian decomposition method needs no discretisation, linearization, transformation or perturbation. In this paper, our aim is to determine the accuracy and efficiency of the Adomian decomposition method in solving initial value problems of linear and non-linear fractional differential equations.

Riemann-Liouville fractional integral

Definition

Suppose that $f(x) \in C([a,b]), \quad a < x < b$ then the Riemann-Liouville fractional integral of order α of a function $f(x)$ is defined by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \cdots\cdots (1)$$

where $\alpha \in]-\infty, \infty[$

This formula represents the integral of arbitrary order $\alpha > 0$, but does not permit order $\alpha = 0$ because it formally corresponds to the identity operator.

Riemann-Liouville Fractional Derivative

Recently many models are formulated in terms of fractional derivatives, such as in control processing, viscoelasticity, signal processing and anomalous diffusion.

Definition

The Riemann-Liouville fractional derivative [4] of a function $f(x)$, where $f(x) \in C([a,b]) and \ \ a < x < b$ with fractional order $\alpha, \alpha \in]0,1[$ is defined as

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^\alpha} \, dt \cdots\cdots (2)$$

This is called the Riemann-Liouville fractional derivative of arbitrary order α.
If $0 < \alpha < 1$ i.e., $\alpha \in]0,1[$
then $D_{a+}^\alpha f(x)$ exists for all $f \in C'([a,b])$ all $x \in]a,b[$

Lemma 1:
for $\alpha, \beta \geq 0, \ f(x) \in \ L_1[0,T]$ then
$I_{0^+}^\alpha I_{0^+}^\beta f(x) = I_{0^+}^{\alpha+\beta} f(x) = I_{0^+}^\beta I_{0^+}^\alpha f(x)$
is satisfied almost everywhere on $[0,T]$. Moreover, if $f(x) \in \ L_1[0,T]$ then the above equation is true for all $x \in [0,T]$.

Caputo Fractional Derivative

Definition

Mathematically [1,2,3] it is defined as,
Suppose that, $\alpha > 0,\ x > a,\ \alpha, a, x\ \in\ \mathbb{R}$

$$D_*^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\, dt, & n-1 < \alpha < n \in N \\ \frac{d^n f(x)}{dx^n} & \alpha = n\ \in N \\ I_{a^+}^{\alpha}\ f(x) & \alpha \leq 0 \end{cases} \quad \cdots\cdots\cdots\cdots (3)$$

is called the Caputo fractional derivative or Caputo fractional differential operator of α.

Lemma 2:

$$\text{If}\ \ \alpha > 0,\ f(x) \in L_1[0,T].$$
$$\text{then,}\ \ {}^{c}D_{0+}^{\alpha}\ I_0^{\alpha}\ f(x) = f(x)\ \ \text{for all}\ \ x \in [0,T]$$

Proof:
By the definition of Riemann-Liouville fractional derivative, by using equation (2)

$$D_{0^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}}\, dt$$

by substituting $\alpha = n, f(x) = x \Rightarrow f(t) = t$
$(x - t) = u$
$\Rightarrow -dt = du$
Also, new limit point will be
when $t = x \Rightarrow u = 0$
when $t = 0 \Rightarrow u = x$

$\therefore$ The above equation becomes,

$$\begin{aligned} D_*^{\alpha} &= \frac{1}{\Gamma(1-n)} \frac{d}{dx} \int_x^0 \frac{(x-u)}{(u)^n}\, (-du) \\ &= \frac{1}{\Gamma(1-n)} \frac{d}{dx} \int_0^x \frac{(x-u)}{(u)^n}\, (du) \\ &= \frac{1}{\Gamma(1-n)} \frac{d}{dx} \int_0^x (x-u)\ u^{-n}\, du \end{aligned}$$

Now, using integration by parts,

$$
\begin{aligned}
D_*^{\alpha} f(x) =& \frac{1}{\Gamma(1-n)} \frac{d}{dx} \int_0^x \left[(x-u) \int_0^x \ u^{-n} du - \left(\int_0^x \frac{d(x-u)}{du} \int_0^x \ u^{-n} du \right) du \right] \\
=& \frac{1}{\Gamma(1-n)} \ \frac{d}{dx} \ \int_0^x \ \frac{u^{(1-n)}}{(-n+1} \ (du) \\
=& \frac{1}{(-n+1)\Gamma(1-n)} \ \frac{d}{dx} \ \left[\frac{u^{-n+1+1}}{-n+1+1} \right]_0^x \\
=& \frac{1}{(-n+1)\Gamma(1-n)} \ \frac{(-n+2)x^{-n+1}}{(-n+2)}
\end{aligned}
$$

Hence

$$D_*^{\alpha} f(x) = \frac{x^{-n+1}}{(-n+1)\Gamma(1-n)} \cdots\cdots (4)$$

Secondly, by the definition of Riemann-Liouville fractional integral of a function $f(x)$

$$I_{0^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$$

Let us put $\alpha = n, f(x) = x \Rightarrow f(t) = t$
$(x-t) = u$
$\Rightarrow -dt = du$
when $t = x \Rightarrow u = 0$
when $t = 0 \Rightarrow u = x$

$\therefore$ The above integral becomes,

$$\begin{aligned}
I_{0^+}^n(x) =& \frac{1}{\Gamma(n)} \int_0^x \frac{x-u}{u^{1-n}} du \\
=& \frac{1}{\Gamma(n)} \int_0^x (x-u)\ u^{n-1} du \\
=& \frac{1}{\Gamma(n)} \left[\frac{(x-u)u^n}{n} \right]_0^x \ - \ \int_0^x \left(\frac{d(x-u)}{du}\ \frac{u^n}{n} \right) du \\
=& \frac{1}{n(n+1)\Gamma(n)}\ x^{n+1} \\
I_{0^+}^n(x) =& \frac{1}{(n+1)\Gamma(n+1)}\ x^{n+1} \qquad \cdots\cdots (5)
\end{aligned}$$

Now,
Multiplying the Riemann-Liouville derivative operator to the equation (5) and using (4), we get

$$\begin{aligned}
D_{0^+}^n I_{0^+}^n(x) =& D_{0^+}^n \left[\frac{x^{n+1}}{(n+1)\Gamma(n+1)} \right] \\
=& D_{0^+}^n \left[\frac{x^n x}{(n+1)\Gamma(n+1)} \right] \\
=& \frac{x^n}{(n+1)\Gamma(n+1)}\ (D_{0^+}^n x) \\
=& \frac{x^n}{(n+1)\Gamma(n+1)}\ \frac{x^{-n+1}}{(1-n)\Gamma(1-n)}
\end{aligned}$$

Hence, $\quad D_{0^+}^n I_{0^+}^n(x) = f(x)$

Adomian Decomposition Method

Consider the differential equation

$$Lv + Rv + Nv = g \cdots\cdots (6)$$

where,
L - highest order derivative & easily invertible.
R - linear differential operator of order less than L.
Nv - represents the non-linear terms.
g - source term.
The functions $v(x)$ is supposed to be bounded for all $x \in I = [0, T]$ & the nonlinear term Nv satisfies Lipschitz condition i.e.,
for initial value problem, we conveniently define L^{-1} for $L = \frac{d^n}{dt^n}$ as the n-fold definite integral from zero to x. If L is second order operator, L^{-1} is a two fold integral & so by solving for v.
$|Nv - Nw| \leq k_1 |v - w|$
where k_1 is a positive constant.
Since L is invertible there, we get,

$$L^{-1}Lv = L^{-1}g - L^{-1}Rv - L^{-1}Nv + B \cdots\cdots\cdots (7)$$

where,
B is the constant of integration & satisfies $LB = 0$.

$$L^{-1}(\cdot) = \int_0^x (\cdot)\ dx$$

Now, the Adomian decomposition method consists of approximating the solution of (6) as an infinite series

$$v(y, x) = \sum_{n=0}^{\infty} v_n(y, x) \cdots\cdots\cdots (8)$$

and the nonlinear term Nv will be decomposed by the infinite series of Adomian polynomials.

$$Nv = \sum_{n=0}^{\infty} An(v_0, v_1, v_2, \cdots, v_n) \cdots\cdots\cdots (9)$$

where An is Adomian polynomial calculated by using the formula,

$$An = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\ N(w(\lambda))\right]_{\lambda=0}, \quad n = 0, 1, 2, \cdots$$

where,

$$w(\lambda) = \sum_{n=0}^{\infty} \lambda^n v_n$$

Substituting the decomposition series i.e. equation (8) & (9) in (7). We get,

$$\sum_{n=0}^{\infty} v_n(y,x) = B + L^{-1}g - L^{-1}R\left(\sum_{n=0}^{\infty} v_n(y,x)\right) - \sum_{n=0}^{\infty} A_n(v_0, v_1 \cdots v_n) \cdots (10)$$

from the above equation, we can say that,

$$v_0 = B + L^{-1}g$$

$$v_1 = -L^{-1}(Rv_0) - L^{-1}(A_0)$$

$$v_2 = -L^{-1}(Rv_1) - L^{-1}(A_1)$$

.

.

.

$$v_{n+1} = L^{-1}(Rv_n) - L^{-1}(A_n), \quad n \geq 0$$

where B is the initial condition.
Hence all terms of V are can be found & the general solution obtained by using Adomian decomposition method as

$$v(y,x) = \sum_{n=0}^{\infty} v_n(y,x) \cdots\cdots (11)$$

The convergence of the series [5]has been proved.
Now, we apply Adomian decomposition method to derive the solution of fractional partial differential equations. We solve few examples by Adomian decomposition method.
Firstly, we apply the Adomian decomposition method to obtain approximate solution of initial value problems for fractional BBM-Burgers equation with $\epsilon = 1$.

Example 1:
Consider the following equation

$$\frac{\partial^\alpha v(y,x)}{\partial x^\alpha} = \Big(L_{yy}\ v(y,x)\Big) - \Big(v(y,x)L_y v(y,x)\Big)$$

where

$$L_{yy} = \frac{\partial^2}{\partial y^2}, \quad L_y = \frac{\partial}{\partial y}$$

with initial condition

$$v(y,0) = \sin y, \quad (y,x) \ \in [0,1] \times [0,T].$$

Solution:
The given equation is

$$\frac{\partial^\alpha v(y,x)}{\partial x^\alpha} = \Big(L_{yy}\ v(y,x)\Big) - \Big(v(y,x)L_y v(y,x)\Big) \cdots\cdots (12)$$

with initial condition

$$v(y,0) = \sin y, \quad (y,x) \ \in [0,1] \times [0,T].$$

and the fractional differential operator $\frac{\partial^\alpha}{\partial x^\alpha}$ defined in the definition of Caputo fractional derivative. Let I^α be the inverse of the operator $\frac{\partial^\alpha}{\partial x^\alpha}$
Now applying I^α to the both sides of given equation (12)
we get,

$$I^\alpha \left[\frac{\partial^\alpha}{\partial x^\alpha} v(y,x) \right] = I^\alpha(L_{yy}v) - I^\alpha(vL_y v) + B$$

$$\Rightarrow v(y,x) = I^\alpha(L_{yy}v) - I^\alpha(Nv) + B$$

where $Nv = vL_y v$
& B is constant of integration.
In order to solve our problems we must generalize these Adomian polynomials as follows.

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n}\ N\Big(\sum_{i=0}^{\infty} \lambda^i v_i\Big)\Big(\frac{\partial}{\partial y}\ \sum_{i=0}^{\infty}\ \lambda^i v_i\Big)\right]_{\lambda=0}, \quad n \geq 0$$

The first few terms of the Adomian polynomials are derived as follows

$$
\begin{aligned}
A_0 =& v_0 \ \frac{\partial v_0}{\partial y} \\
A_1 =& \frac{1}{1!}\left[\frac{d}{d\lambda}\left[(v_0+\lambda v_1)\ \left(\frac{\partial v_0}{\partial y}+\frac{\lambda \partial v_1}{\partial y}\right)\right]\right] \\
A_1 =& v_0\frac{\partial v_1}{\partial y}\ +\ v_1\frac{\partial v_0}{\partial y_1} \\
A_2 =& \frac{1}{2!}\left[\frac{d^2}{d\lambda^2}\left[(v_0+\lambda v_1+\lambda^2 v^2)\left(\frac{\partial v_0}{\partial y}+\frac{\lambda \partial v_1}{\partial y}+\ \frac{\lambda^2 \partial v_2}{\partial y}\right)\right]_{\lambda=0}\right. \\
A_2 =& v_0\frac{\partial v_2}{\partial y}\ +\ v_1\frac{\partial v_1}{\partial y}\ + v_2\frac{\partial v_0}{\partial y}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
A_3 =& v_0\frac{\partial v_3}{\partial y}\ +\ v_1\frac{\partial v_2}{\partial y}\ + v_2\frac{\partial v_1}{\partial y} + v_3\ \frac{\partial v_0}{\partial y} \\
&\ \ \ \cdot \\
&\ \ \ \cdot \\
&\ \ \ \cdot
\end{aligned}
$$

and so on

Using equation (10)we get

$$
v(y,x) = B + I^{\alpha}\ \left(\sum_{n=0}^{\infty}(L_{yy}v_n)\right) - I^{\alpha}\ \left(\sum_{n=0}^{\infty}(A_n)\right)
$$

This gives,

$$
\begin{aligned}
v_0 =& B = v(y,0) \\
v_1 =& I^{\alpha}(L_{yy}v - 0) - I^{\alpha}A_0 \\
v_2 =& I^{\alpha}(L_{yy}v - 1) - I^{\alpha}A_1 \\
& . \\
& . \\
& . \\
v_{n+1} =& I^{\alpha}(L_{yy}v - n) - I^{\alpha}A_n
\end{aligned}
$$

by putting the values of $v_0, v_1, \cdots$ from above we get the solution of the initial value problem.

$$v(y,x) = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$$

we have the initial conditions, using this we get

$$
\begin{aligned}
v_0 =& v(y,0) = f(y) = \sin y \\
v_1 =& I^{\alpha}(L_{yy}v - 0) - I^{\alpha}A_0 \\
v_1 =& \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-t)^{1-\alpha}} f''(y)dt - \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-t)^{1-\alpha}} f(x)\, dx \\
v_1 =& \frac{f_1(y)}{\Gamma(\alpha+1)}\, x^{\alpha}
\end{aligned}
$$

where,

$$\begin{aligned} f_1(y) &= -f''(y) + f(y)f'(y) \\ &= \sin y + \sin y \cdot \cos y \\ &= \sin y + (1 + \cos y) \\ v_2 =& I^\alpha(L_{yy}v - 1) - I^\alpha A_1 \\ &= f_2(y)\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} \end{aligned}$$

where

$$\begin{aligned} f_2(y) &= f_1''(y) - f(y)f_1'(y) - f_1(y)\cdot f'(y) \\ &= \sin^3 y + [-1 - 5\cos y - \cos^2 y - (1+\cos y)\cos y]\cdot \sin y \end{aligned}$$

$$\begin{aligned} v_3 =& I^\alpha(L_{yy}v_2) - \ I^\alpha(A_2) \\ v_3 =& f_3(y)\ \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} \\ & Similarly, \\ v_4 =& \frac{f_4(y)}{\Gamma(4\alpha+1)}\ x^{4\alpha} \\ & . \\ & . \\ & . \\ v_n =& \frac{f_n(y)}{\Gamma(n\alpha+1)}\ x^{n\alpha} \end{aligned}$$

$\therefore$ Summing all these terms i.e., the solution of the equation in series is given by

$$v(y,x) = f(y) + \frac{x^{\alpha}}{\Gamma(\alpha+1)} f_1(y) + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} f_2(y) + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} f_3(y) + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} f_4(y) + \cdots + \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} f_n(y) + \cdots$$

$$v(y,x) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} f_n(y)$$

where $f_0(y)$ is an initial condition.

Example 2:
Consider the following nonlinear fractional equation:

$$D_x^{\alpha} v \ + \ D_y^2 v \ - \ D_y v \ + \ v^2 = 0$$

$$0 < y \leq 1, \ 0 \leq x \leq 1, \ 0 < \alpha \leq 1$$

with initial condition,

$$v(y,0) = \phi = f(y) = u^2$$

$$(y,x) \in \Omega \times [0,T]$$

here $\Omega = (0,1)$
Solution:
Given nonlinear fractional equation is

$$D_x^{\alpha} v \ + \ D_y^2 v \ - \ D_y v \ + \ v^2 = 0 \cdots\cdots (13)$$

The standard form of the given fractional equations operator form is

$$D_x^{\alpha} v = -[v(y,x)]^2 - L_{yy} v(y,x) + L_y v(y,x) \cdots\cdots (14)$$

where,

$$L_{yy} = \frac{\partial^2}{\partial y^2}, \;\; L_y = \frac{\partial}{\partial y}$$

& the fractional operator $D_x^\alpha v$ defined in the definition of Caputo fractional derivative respectively.
I^α is the inverse of D_x^α
Now,
applying I^α to the both side of equation (14)

$$v(y,x) = -I^\alpha(Nv) - I^\alpha\big(L_{yy}v(y,x)\big) + I^\alpha\big(L_y v(y,x)\big) + \phi$$

where $Nv = v^2$.
According to the Adomian decomposition method we assume series solution for the unknown function $v(y,x)$ in the form

$$v(y,x) = \sum_{n=0}^{\infty} v_n(y,x)$$

To solve this problem, we must genralize these Adomian polynomials as follows

$$\begin{aligned}
A_n =& \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\sum_{i-0}^{\infty} \lambda^i v_i \right) \left(\sum_{i=0}^{\infty} \lambda^i v_i \right) \right]_{\lambda=0} \\
\therefore A_0 =& v_0 \cdot v_0 = v_0^2 \\
A_1 =& \frac{d}{d\lambda} \big[(v_0 + \lambda v_1)(v_0 + \lambda v_1) \big]_{\lambda=0} \\
A_1 =& v_0 v_1 + v_1 v_0 \\
A_1 =& 2 v_0 v_1
\end{aligned}$$

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\Big[\big(v_0 + \lambda v_1 + \lambda^2 v_2\big)(v_0 + \lambda v_1 + \lambda^2 v_2)\Big]_{\lambda=0}$$
$$= 2v_0 v_2 + v_1^2$$
$$A_3 = \frac{1}{3!}\frac{d^3}{d\lambda^3}\Big[\big(v_0 + \lambda v_1 + \lambda^3 v_3\big)(v_0 + \lambda v_1 + \lambda^2 v_2 + \lambda^3 v_3)\Big]_{\lambda=0}$$
$$= 2v_0 v_3 + 2v_1 v_2$$

.

.

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and so on.

Now, using equation (10) by Adomian decomposition method, we have,

$$\sum_{n=0}^{\infty} v_n(y,x) = -I^{\alpha}\big(\sum A_n(v_0, v_1, \cdots, v_n)\big) - I^{\alpha}\big(\sum_{n=0}^{\infty}(L_{yy}v_n)\big) + I^{\alpha}\big(\sum_{n=0}^{\infty}(L_y v_n)\big) + \phi$$

from this equation we observe that,

$$v_0 = v(y,0) = f(y)$$
$$v_1 = -I^{\alpha}(A_0) - I^{\alpha}(L_{yy}v_0) + I^{\alpha}(L_y v_0)$$
$$v_2 = -I^{\alpha}(A_1) - I^{\alpha}(L_{yy}v_1) + I^{\alpha}(L_y v_1)$$

.

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.

$$v_{n+1} = -I^{\alpha}(A_n) - I^{\alpha}(L_{yy}v_n) + I^{\alpha}(L_y v_n)$$

by substituting the values of $v_0, v_1, v_2, \cdots$
we get the solutions of the initial value problems

$$\begin{aligned}
v =& v_0 + v_1 + v_2 + v_3 + \cdots + v_n + \cdots \\
v_0 =& v(y,x) = f(y) = y^2 \\
v_1 =& -I^\alpha(A_0) - I^\alpha(L_{yy}v_0) + I^\alpha(L_y v_0) \\
=& \frac{x^\alpha}{\Gamma(\alpha+1)} f_1(y)
\end{aligned}$$

where,

$$\begin{aligned}
f_1(y) &= \big(f(y)\big)^2 + f''(y) - f'(y) \\
&= y^4 - 2y + 2
\end{aligned}$$

$$\begin{aligned}
\therefore v_1 =& y^4 - 2y + 2\left[\frac{x^\alpha}{\Gamma(\alpha+1)}\right] \\
v_2 =& -I^\alpha(A_1) - I^\alpha(L_{yy}v_1) + I^\alpha(L_y v_1) \\
=& \frac{x^{2\alpha}}{\Gamma(2\alpha+1)}\ f_2(y)
\end{aligned}$$

where,

$$\begin{aligned}
f_2(y) &= -[2f(y)f_1(y) + f_1''(y) - f_1'(y)] \\
&= -2y^6 + 8y^2 + 2
\end{aligned}$$

also

$$\begin{aligned}
v_3 =& -I^\alpha(A_2) - I^\alpha(L_{yy}v_2) + I^\alpha(L_y v_2) \\
v_3 =& \frac{x^{3\alpha}}{\Gamma(3\alpha+1)}\ f_3(y)
\end{aligned}$$

where

$$f_3(y) = -2f(y)f_2(y) - 2f_1^2(y) - f_2''(y) + f_2'(y)$$
$$= 3y^8 - 8y^5 + 40y^4 - 8y^2 + 24y - 20$$

Similarly,

$$v_4 = \frac{x^{4\alpha}}{\Gamma(4\alpha+1)}\ f_4(y)$$

$$.$$

$$.$$

$$.$$

$$v_n = \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}\ f_n(y)$$

$\therefore$ summing all these terms
$\therefore$ The solution of the given initial value problem is given by,

$$v(y,x) = f(y) + \frac{x^{\alpha}}{\Gamma(\alpha+1)}\ f_1(y) + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} f_2(y) + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} f_3(y) + \cdots + \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} f_n(y)$$

$$\therefore v(y,x) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} f_n(y)$$

this gives the solution of initial value problem.

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