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MEASURABLE FUNCTION AND LEBESGUE MEASURE

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ABSTRACT :
Measurable function introduce including 1-dimensional Lebesgue measure as the primary definitions, example and develop simple properties of them. The Lebesgue measure of the interval [0,1] in the real number is its length in the everyday sense of the word, specifically 1. In this paper we present also the Essential Supremum, Essential Infimum and some theorem in Measurable function.

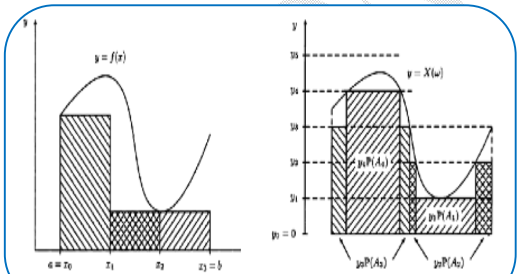


Fig. 1.3.1. Lower Riemann sum. Fig. 1.3.2. Lower Lebesgue sum.

KEYWORDS : *Measurable set, Lebesgue Measure, Borel set, σ – algebra, Essential Supremum. Essential Infimum.*

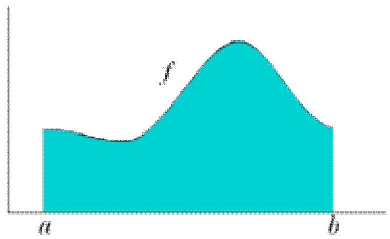
INTRODUCTION

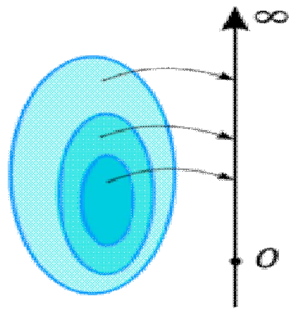
Measure theory was initially created to provide a useful abstraction of the notion as length of subsets of the real line and more generally, area and volume of subsets of Euclidian spaces. In particular, it provided a systematic answer to the equation of which subset of $\mathbb{R}$ have a length. The term Lebesgue integration can be mean either the general theory of integration of a function with respect to a general measure, as introduced by Lebesgue of the specific case of integration of a function defined on a sub domain of the real line with respect to the Lebesgue measure.

DEFINITION: MEASURE

Let A be σ – algebra a function $\mu: \mathcal{A} \rightarrow \mathbb{R} \cup \{\infty\}$ is called a Measure, if

a. $\mu(\Phi) = 0$
b. $\mu(A) \geq 0$ for all $A \in \mathcal{A}$, $\mu(\cup A_k) = \sum \mu(A_k)$ for any sequence A_k of point wise disjoint sets from A that is $A_i \cap B_j = \Phi$ for $i \neq j$.
c. For any sub set $A \epsilon \mathcal{A}$ with $\mu(A) = \infty$, there exits $B \in \mathcal{A}$ such that $B \subset A$, and $0 < \mu(B) < \infty$.





Lebesgue integral *Measure*

Definition: Measure space

We consider a fixed but arbitrary σ – algebra with a measure. If A is σ – algebra of subsets of X and μ is a measure on A then the triple (X, A,μ) is called a measure space. The sets belonging to A are called measurable sets because the measure is defined for them.

Definition: σ – algebra

Let G contain are equal to P(X), then the set of all σ – algebra containing G is nonempty since it contains P(X). Hence we talk about the minimal σ – algebra containing G. This σ – algebra is called the σ – algebra generated by G.

Properties

1) Extends length: for every interval I, μ(I) = 1.
2) Monotone: If $A \subset B \subset \mathbb{R}$, then $0 \le \mu(A) \le \mu(B) \le \infty$.
3) Translation Invariant: For each subset A of $\mathbb{R}$ and for each point $x_0 \in \mathbb{R}$ we define $A + x_0 = \{x + x_0;\ x \in A\}$. Then $\mu(A + x_0) = \mu(A)$.
4) Countable additive: If A and B are disjoint sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$
5) If $\{A_n\}$ is a sequence of disjoint sets, then

$\mu\left(\bigcup_{n=1}^{\infty}(A_n)\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Example

Show that if $F \in M$ and $\mu(F \Delta G) = 0$ then G is Measurable.

Solution

Given that $\mu(F \Delta G) = 0 \Leftrightarrow F \Delta G$ is measurable.

$\Leftrightarrow (F - G) \cup (G - F)$ is measurable

$\Leftrightarrow (F - G)$ and $(G - F)$ is measurable

Clearly, $F \cap G = F - (F - G)$

Since F and F – G is measurable, and then the difference is also measurable

Hence $F \cap G$ is Measurable.

Example

Show that every nonempty open set has positive Measure.

Solution

Let G be a nonempty open set with disjoint interval.

Such that $G = \bigcup_{n=1}^{\infty} I_n$

$$\mu(G) = \mu\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$= \sum_{n=1}^{\infty} \mu(I_n)$$

$$= \sum_{n=1}^{\infty} I(I_n)$$

Therefore, every nonempty open set has positive Measure.

Definition: Borel set

The σ – algebra generated by the class of interval of the form [a,b) its members are called the Borel set of ℝ. It is denoted by $\boldsymbol{\mathcal{B}}$.

Definition: Borel function

A function f is Borel measurable or Borel function if for all α, {x; f(x) > α} is a Borel set.

Definition: Lebesgue outer measure

Given a subset E ⊂ ℝ within the length of interval I = [a, b] or I = (a, b) given by (I) = b – a. the Lebesgue outer measure λ* (E) is defined as

$$\lambda^*(E) = \inf \left\{ \sum_{i=0}^{n} \ell(I_k);\ (I_k)\ k \in N \right.$$

Definition: Lebesgue measure

If E is a Lebesgue measurable set, then the Lebesgue measure of E is defined to be its outer measure λ* (E) and is written by $\lambda(E)$.

Definition: Lebesgue measurable

A set E ⊂ ℝ is called Lebesgue measurable if for every subset A of ℝ,
$\lambda^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \overline{E})$

Definition: Lebesgue measurable function

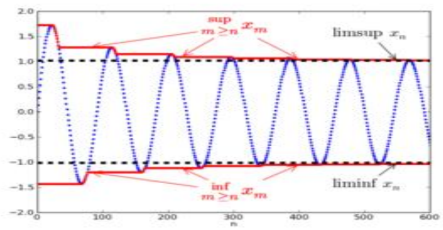
Let f be an extended real value function defined on a measurable set E, then f is a Lebesgue measurable function if for each a ∈ ℝ, the set { x ; f(x) > α} is measurable.

Definition: Almost everywhere (a.e)

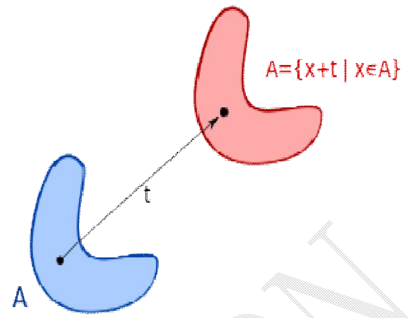
It a property holds except on a set of measure zero. We say it holds almost everywhere is usually denoted by (a.e).

Properties

1) If A is a Cartesian product of intervals $I_1 \times I_2 \times \ldots \times I_n$ then A is Lebesgue measurable and $\lambda(A) = |I_1| \cdot |I_2| \cdot |I_3| \ldots |I_n|$
2) If A is Lebesgue measurable, then so is its complement.
3) If A and B are Lebesgue measurable and A is a subset of B then $\lambda(A) \leq \lambda(B)$.
4) $\lambda(A) \geq 0$ for every Lebesgue measurable set A.
5) Countable union and intersection of Lebesgue measurable sets are Lebesgue measurable sets.
6) If A is open (or) closed subset of R^n, then A is Lebesgue measurable.



Essential Supremum & Essential Infimum



Lebesgue measurable

Example

1. Any open or closed interval [a, b] of real numbers is Lebesgue measurable and its Lebesgue measure of the length b – a, the open interval (a, b) has the same measure. Since the difference between two set consists only of the end points a and b has *measure zero*.
2. Any Cartesian product of intervals [a, b] and [c, d] is Lebesgue measurable, and its Lebesgue measure is (b - a)(d - c), then area of the corresponding *Rectangle*.
3. Let $\{f_n\}$ be a sequence of measurable function Almost everywhere to f, then f is measurable. Since f = lim sup f.

Example

If f is a measurable function then so that $f^+ = \max\{f,0\}$ and $f^- = \max\{f,0\}$

Solution

$f^+ = \{ f, \text{ if } f > 0\}$

$f^+ = \{ f, \text{ if } f \leq 0\}$

If $f > 0$ then $f^+ = f$, therefore f^+ is a measurable. if $f \leq 0$ then $f^+ = 0$ where 0 is a constant.

Since constant function is measurable, then f^+ is a measurable. Similarly we can prove f^- is a measurable.

Example

The set of points on which a sequence of measurable function $\{ f_n\}$ convergence is measurable.

Solution

Let A be the set of points on which a sequence of measurable function convergence. That is A = $\{x; \limsup f_n(x) = \liminf f_n(x) = 0\}$ Clearly A is measurable.

Essential Supremum

Let f be measurable function then $\inf\{ \alpha \,;\, f \leq \alpha, \text{(a.e)}\}$is called Essential Supremum of f.

Essential Infimum.

Let f be measurable function then, $\inf\{ \alpha \,;\, f \geq \alpha, \text{(a.e)}\}$is called Essential Infimum of f.

Example

Prove that Ess.sup f = - Ess.sup (-f)

Solution

Ess.sup f = inf {α; f ≤ α, (a.e)}
= inf {α; - f ≥ - α, (a.e)}
= -sup {- α ; f ≥ - α, (a.e) }
= - Ess.sup (-f)

Theorem

For any measurable function f and g, ess.sup (f+g) ≤ ess.sup(f) + ess.sup(g)

Proof

We know that, f ≤ ess ·sup(f) (a.e)
g ≤ ess ·sup(g) (a.e)
f + g ≤ ess ·sup(f) (a.e) + ess ·sup(g) (a.e)

f and g are measurable. ess. sup (f+g) ≤ ess· sup(f) + ess· sup(g).

Theorem

The following statements are equivalent

i. f is measurable function
ii. For all α, {x; f(x) ≥ α} is measurable.
iii. For all α, {x; f(x) < α} is measurable.
iv. For all α, {x; f(x) ≤ α} is measurable.

Proof:

(i) = (ii)

Assume that f is measurable, therefore {x; f(x) > α} is measurable.

$\{x; f(x) \geq \alpha\} = \cap_{n=1}^{\infty}\{ x; f(x) > \alpha - \frac{1}{n}\}$

Therefore RHS is measurable and hence LHS is also measurable.

(ii) = (iii)

{x; f(x) < α} = c {x; f(x) ≥ α} by (ii), {x; f(x) ≥ α} is measurable.

Therefore c {x; f(x) ≥ α} is measurable.

Therefore RHS is measurable and LHS is measurable.

(iii) = (iv)

Assume that for all α, {x; f(x) < α} is measurable.

Here $\{x; f(x) \leq \alpha\}$ and $\cap_{n=1}^{\infty}\{ x; f(x) > \alpha - \frac{1}{n}\}$

Each set in the RHS is measurable.

Hence its intersection is also measurable. Therefore LHS is measurable.

Theorem

Let E be a measurable set then for each Y. then E + Y = {x + y; x ∈ E} is measurable and the measures are same.

Proof

Given that E is measurable then there exits as open set O such that E ⊂O,

μ (O – E) ≤ ∈ => E (because is measurable). Clearly O + Y is also open, E + Y ≤ O + Y

Hence, O – E = (O + Y) – (E + Y)
O – E = (O – E) + Y
μ (O – E) = μ [(O – E) + Y]

$\mu\,[(O - E) + Y] \leq \in$

$\mu\,[O - (E - Y)] \leq \in$

Therefore E + Y are measurable.

$\mu^*(A) = \mu^*(A + x)$

$\mu(A) = \mu(A + x)$

E and E +Y have same measure.

CONCLUSION

The concept of measurable function defined on measurable space with values in extended real line some of its basic properties, Examples has been discussed in this paper. This led to the introduction of Lebesgue measure, its basic properties, Borel function on the real line are presented.

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