REVIEW OF RESEARCH
IMPACT FACTOR : 5.7631(UIF) UGC APPROVED JOURNAL NO. 48514 ISSN: 2249-894X
VOLUME - 8 | ISSUE - 4 | JANUARY - 2019

ON FRACTIONAL INTEGRAL OPERATORS ASSOCIATED WITH ALEPH-FUNCTION FOR REAL POSITIVE SYMMETRIC DEFINITE MATRIX

Yashwant Singh
Department of Mathematics,
Government College, Kaladera, Jaipur, Rajasthan, India.

ABSTRACT :

In the present paper, the author has defined fractional integral operators associated with Aleph($\aleph$)-function for real positive symmetric definite matrix . Some special cases of our operators have been mentioned.

KEYWORDS : *Fractional Integral Operators, $\aleph$ -function, H -function, Matrix transform, Symmetric matrix*
(2010 Mathematical Subject Classification : 33C99)

INTRODUCTION:

(a) $\aleph$ -function with matrix argument

Let X is a $p \times p$ real symmetric positive definite matrix of functionally independent variables. Let the $\aleph$ - function introduced by Suland et.al. [7] defined and represented in the following form:

$$\aleph[z] = \aleph^{m,n}_{p_i,q_i;\tau_i;r}[z] = \aleph^{m,n}_{p_i,qi;\tau_i;r}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix}\right] = \frac{1}{2\pi\omega}\int_L \theta(s) z^s ds \tag{1.1}$$

Where $\omega = \sqrt{-1}$;

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s\right\}} \tag{1.2}$$

We shall use the following notations:

$A^* = (a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}; B^* = (b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}$

It is assumed that $\aleph(XY) = \aleph(YX)$ for real symmetric $m \times m$ positive definite matrices X and Y, $\aleph(X)$ is defined by the following integral equation:

$$\int_{X>0} |X|^{\rho-\frac{m+1}{2}} \aleph(X)dX = \frac{\prod_{j=1}^{m}\Gamma_m(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma_m(\frac{m+1}{2}-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma_m(\frac{m+1}{2}-b_{ji}+\beta_{ji}\xi)\prod_{j=N+1}^{p_i}\Gamma_m(a_{ji}-\alpha_{ji}\xi)\right\}} \quad (1.3)$$

(b) Matrix transform

A generalized matrix transform or M-transform of a function $f(X)$ of a $m\times m$ real symmetric positive definite or strictly negative definite matrix X is defined as follows:

$$M_f(s) = \int_{X>0} |X|^{s-\frac{m+1}{2}} f(X)dX \qquad (X>0) \qquad (1.4)$$

Whenever $M_f(s)$ exists. Also $f(X)$ is assumed to be a symmetric function i.e. $f(BX)=f(XB)=f\left(B^{1/2}XB^{1/2}\right)$ for $B=B'>0$. When $X<0$ replace X by $-X$ in M-transform.

(c) Integral operators involving $\aleph$-function

$Y[f(X)] = Y\left[f(X)\middle|\sigma,\rho,\gamma;{}_{B^*}^{A^*}\right]$ =

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}\aleph\left[\gamma(1-UX^{-1})\Big|_{B^*}^{A^*}\right]f(U)dU \qquad (1.5)$$

$N[f(X)] = N\left[f(X)\middle|\delta,\rho,\gamma;{}_{B^*}^{A^*}\right]$ =

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}\aleph\left[\gamma(I-XU^{-1})\Big|_{B^*}^{A^*}\right]f(U)dU \qquad (1.6)$$

The above defined operators exists under the following conditions:

(i) $p_i\ge 1, q_i<\infty, \frac{1}{p_i}+\frac{1}{q_i}=1, |\arg(I-a)|<\pi$, (ii) ($\mathrm{Re}(\sigma)>\frac{1}{q_i}, \mathrm{Re}(\delta)>\frac{1}{p_i}, \mathrm{Re}(\rho)>\frac{m+1}{2}$

(iii) $\mathrm{Re}(\alpha+\min_{1\le j\le M}\tau_i\frac{b_j}{\beta_j})>\frac{m+1}{2}$ (iv) $f(X)\in L_P(0,\infty)$.

The last condition ensures that $Y[f(X)]$ and $N[f(X)]$ both exist and also both belong to $L_P(0,\infty)$.

2. Main Results

The following theorems of the operators defined by (1.5 and (1.6) have been established in the expression of matrix transform:

Theorem 1: If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where

$\mathrm{Re}(\alpha + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > -\frac{1}{q_i}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\sigma - t + 1) > \frac{m+1}{2}$ and $|\arg(I-a)| < \pi$ then

$$M\{Y[f(X)]\} = \frac{\Gamma_m\left(\sigma - t + \frac{m+1}{2}\right)}{\Gamma_m(\rho)} \aleph^{m,n+1}_{p_i+1,q_i+1:\tau_i:r}\left[\gamma I \middle| \begin{matrix} \left(\frac{m+1}{2},1;1\right), A^* \\ B^*, \left(\frac{m+1}{2} - t - \sigma - \rho, 1;1\right) \end{matrix}\right] M[f(U)] \qquad (2.1)$$

Where I is $m \times m$ identity matrix.

Proof: Taking the matrix transform of equation (1.5), we get

$$M\{Y[f(X)]\} = \int_{X>0} |X|^{t-\frac{m+1}{2}} \left[\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\right.$$

$$\left.\int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} \aleph\left[\gamma(I-UX^{-1})\middle|^{A^*}_{B^*}\right] f(U)dU\right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\{Y[f(X)]\} = \frac{1}{\Gamma_m(\rho)} \int_{X>0} |U|^{\sigma} f(U)dU$$

$$\left.\int_{0<U<X} |X|^{t-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} \aleph\left[\gamma(I-UX^{-1})\middle|^{A^*}_{B^*}\right]\right] dX$$

On evaluating X-integral with the help of the result :

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} \aleph\left[|XZ|\middle|^{A^*}_{B^*}\right] dX = \Gamma_m(\rho) \aleph^{m,n+1}_{p_i+1,q_i+1:\tau_i:r}\left[|Z| \middle| \begin{matrix} \left(\frac{m+1}{2} - \delta, 1\right), A^* \\ B^*, \left(\frac{m+1}{2} - \delta - \rho, 1\right) \end{matrix}\right]$$

Where $\mathrm{Re}(\alpha + \min_{1\le j\le m} \tau_i \frac{b_j}{\beta_j}) > \frac{m+1}{2}$ and $\mathrm{Re}(\rho) > \frac{m+1}{2} - 1$.

We obtain the required result.

Theorem 2: If $f(X)\in L_P(0,\infty)\ 1\le P\le 2[or f(X)\in M_P(0,\infty)\ and\ P>2]$ where $\mathrm{Re}(\delta+\min\limits_{1\le j\le M}\frac{b_j}{\beta_j})>\frac{m+1}{2}$, $\mathrm{Re}(\delta)>-\frac{1}{q_i},\mathrm{Re}(t)>\frac{m+1}{2},\mathrm{Re}(\delta+t)>\frac{m+1}{2}$ and $|\arg(I-a)|<\pi$ then

$$M\left\{N[f(X)]\right\}=\frac{\Gamma_m(\delta+1)}{\Gamma_m(\rho)}\aleph^{m,n+1}_{p_i+1,q_i+1:\tau_i:r}\left[\gamma I\left|\begin{matrix}\left(\frac{m+1}{2}-\delta,1\right),A^*\\ B^*,\left(\frac{m+1}{2}-\delta-\rho,1\right)\end{matrix}\right.\right]M[f(U)] \qquad (2.2)$$

Where I is $m\times m$ identity matrix.

Proof: Taking the matrix transform of equation (1.6), we get

$$M\left\{N[f(X)]\right\}=\int\limits_{X>0}|X|^{\delta-\frac{m+1}{2}}\left[\frac{|X|^{\delta}}{\Gamma_m(\rho)}\right.$$

$$\left.\int\limits_{U>X}|U|^{-\sigma-\rho}|U-X|^{\rho-\frac{m+1}{2}}\aleph^{m,n}_{p_i,q_i:\tau_i:r}\left[\gamma(I-XU^{-1})\Big|^{A^*}_{B^*}\right]f(U)dU\right]dX \qquad (2.3)$$

And changing the order of integration and evaluation X-integral with the help of (2.3), we obtain the required result.

Theorem 3:

If $f(X)\in L_P(0,\infty),g(X)\in L_P(0,\infty)$ where $\mathrm{Re}(\delta+\min\limits_{1\le j\le M}\frac{b_j}{\beta_j})>\frac{m+1}{2}$,

$\mathrm{Re}(\delta)>-\frac{1}{q_i},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\sigma)>\max\left(\frac{1}{p_i}+\frac{1}{q_i}\right)$ and $|\arg(I-a)|<\pi$ then

$$\int\limits_{X>0}f(X)Y\left[g(X)\big|\sigma,\rho,\gamma;^{A^*}_{B^*}\right]dX=\int\limits_{X>0}g(X)N\left[f(X)\big|\sigma,\rho,\gamma;^{A^*}_{B^*}\right]dX \qquad (2.4)$$

Proof: Equation (2.4) immediately follows on interpreting it with the help of equations (1.5) and (1.6).

Special Cases

(i) If we put $m=1,n=1,p=2,q=2,\gamma=1,\tau_i=1,r=1$, then the operators (1.5) and (1.6) reduce to their Mellin transforms in the following form:

$$Y[f(X)]=\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int\limits_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H^{1,2}_{2,2}\left[(I-UX^{-1})\right]f(U)dU$$

Here

$$Y[f(X)]=Y\left[f(X)\big|\sigma,\rho,1;^{(a_1;\alpha_1;1),(a_2;\alpha_2)_{1,2}}_{(b_1,\beta_1),(b_2,\beta_2;1)_{1,2}}\right]$$

And

$$H_{2,2}^{1,2}\left[(I-UX^{-1})\right]=H_{2,2}^{1,2}\left[(I-UX^{-1})\Big|_{(b_1,\beta_1),(b_2,\beta_2;1)}^{(a_1,\alpha_1;1),(a_2,\alpha_2)}\right]$$

Then

$$Y[f(X)]=\frac{\Gamma_m(\chi_1)|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|I-UX^{-1}|^{\rho-\beta_1-\frac{m+1}{2}}\ {}_2F_1\left[-;(I-UX^{-1})\right]f(U)dU$$

Where

$$\Gamma(\chi_1)=\frac{\Gamma_m\left(\frac{m+1}{2}-\alpha_1+\beta_1\right)\Gamma_m\left(\frac{m+1}{2}-\alpha_2+\beta_2\right)}{\Gamma_m\left(\frac{m+1}{2}-\alpha_1-\alpha_2+\beta_1+\beta_2\right)}$$

$${}_2F_1\left[-;\left(I-UX^{-1}\right)\right]={}_2F_1\left[\frac{m+1}{2}-\alpha_1-\beta_1,\frac{m+1}{2}-\alpha_2-\beta_2;\frac{m+1}{2}-\beta_1-\beta_2;-(I-UX^{-1})\right]$$

By virtue of the result [6].

Taking M -transform on both sides, we get

$$M\left\{Y(f(X)\right\}=\frac{\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\ {}_3F_2(-;I)M[f(U)]$$

Where $$\Gamma(\chi_2)=\frac{\Gamma_m\left(\frac{m+1}{2}+\sigma\right)\Gamma_m(\rho+\beta_1)}{\Gamma_m\left(\sigma+\rho+\beta_1+\frac{m+1}{2}\right)}$$

And

$${}_3F_2(-;I)={}_3F_2\left(\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2,\sigma+\frac{m+1}{2};\right.$$
$$\left.\frac{m+1}{2}-\beta_2+\beta_1,\sigma+\rho+\frac{m+1}{2}+\beta_1;I\right),$$

$$N[f(X)]=\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{\sigma}|U-X|^{\rho-\frac{m+1}{2}}\ H_{2,2}^{1,2}\left[(I-XU^{-1})\right]f(U)dU$$

Where

$$N[f(X)] = N\left[f(X) \middle| \sigma, \rho, 1;^{(a_1;\alpha_1;1),(a_2;\alpha_2)_{1,2}}_{(b_1,\beta_1),(b_2,\beta_2;1)_{1,2}} \right]$$, and

$$H^{1,2}_{2,2}\left[(I - XU^{-1})\right] = H^{1,2}_{2,2}\left[(I - XU^{-1})\Big|^{(a_1,\alpha_1;1),(a_2,\alpha_2)}_{(b_1,\beta_1),(b_2,\beta_2;1)}\right]$$

Then

$$N[f(X)] = \frac{\Gamma_m(\chi_1)|X|^{\delta+\rho-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{-\delta-\rho}|I - XU^{-1}|^{\rho+\beta_1-\frac{m+1}{2}} {}_2F_1\left[-;(I - XU^{-1})\right] f(U)dU$$

Where

$${}_2F_1\left[-;\left(I - XU^{-1}\right)\right] =$$

$${}_2F_1\left[\frac{m+1}{2} - \alpha_1 + \beta_1, \frac{m+1}{2} - \alpha_2 + \beta_2; \frac{m+1}{2} + \beta_1 - \beta_2; -(I - XU^{-1})\right]$$

Taking M -transform on both sides, we get

$$M\left\{N(f(X)\right\} = \frac{(-1)^{\rho-\frac{m+1}{2}}\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)} {}_2F_1(-;I)M[f(U)]$$

(ii) Putting $p = 0, q = 1, m = 1, n = 0, \gamma = 1, \tau_i = 1, r = 1$, then operators (1.5) and (1.6) reduce to their Mellin transform in the following forms:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma}|X - U|^{\rho-\frac{m+1}{2}} H^{1,0}_{0,1}\left[(I - UX^{-1})\Big|^{-}_{(\beta,1)}\right] f(U)dU$$

Where

$Y[f(X)] = Y\left[f(X) \middle| \sigma, \rho, 1;^{-}_{(\beta,1)} \right]$, and $H^{1,0}_{0,1}\left[(I - UX^{-1})\Big|^{-}_{(\beta,1)}\right] = |I - UX^{-1}|^{\beta} e^{-tr(i-UX^{-1})}$

$$= \frac{|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma}|I - UX^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-UX^{-1})} f(U)dU$$

By virtue of the result [7]. Taking M -transform on both sides, we get

$$M\left\{Y(f(X)\right\} = \frac{\Gamma_m(\rho+\beta)}{\Gamma_m(\rho)} M[f(U)]$$

Also

$$N[f(X)] = \frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right] f(U)dU$$

Where

$N[f(X)] = N\left[f(X)\big|\delta,\rho,1;_{(\beta,1)}^{-}\right]$, and

$$H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right] = |I-XU^{-1}|^{\beta} e^{-tr(i-XU^{-1})}$$

$$= \frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\frac{m+1}{2}} |I-XU^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-XU^{-1})} f(U)dU$$

By virtue of the result [3]. Taking M -transform on both sides, we get

$$M\{N[f(X)]\} = \frac{\Gamma_m\left(\rho-\delta+\beta-\frac{m+1}{2}\right)}{\Gamma_m(\rho)} M[f(U)]$$

If we put $\alpha_j = \beta_j = 1; (j=1,...,P; j=1,...,Q)$ the operators reduce to G -function given by Vyas [8].

Theorem4. If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where $\mathrm{Re}(\alpha) > \frac{m+1}{2}$,

$\mathrm{Re}(\sigma) > -\frac{1}{q}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\sigma) > \frac{m+1}{2}$ and $|\arg(I-a)| < \pi$ then

$$M\left\{R[_{\sigma,\rho,a}^{\alpha}; f(X)]\right\} = \frac{\Gamma_m\left(\sigma-s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\sigma-s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$${}_1F_1\left[\rho+\alpha;\sigma-s+\alpha+\rho+\frac{m+1}{2};I\right] M[f(U)] \qquad (2.5)$$

Proof: Using the Mellin transform of

$R[f(X)] = R\left[_{\sigma,\rho,a}^{(a_r),(b_r)}; f(X)\right] =$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} G_{r,s}^{p,q}\left[a(I-UX^{-1})\Big|_{(b_r)}^{(a_r)}\right] f(U)dU \qquad (2.6)$$

We get

$$M\left\{R[^{\alpha}_{\sigma,\rho,a};f(X)]\right\}=$$

$$\int\limits_{X>0}\frac{|X|^{s-\frac{m+1}{2}}|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\left[\int\limits_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{-}_{(b_1)}\right]f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$\int\limits_{0<U<X}|X|^{s-\sigma-\rho-\frac{m+1}{2}}|X-U|^{\rho-\frac{m+1}{2}}G^{1,0}_{0,1}\left[a(I-UX^{-1})\right]\Bigg]dX=\frac{1}{\Gamma_m(\rho)}\int\limits_{U>X}|U|^{\sigma}f(U)dU$$

$$\int\limits_{X>U}|X|^{S-\sigma-\rho-\alpha-\frac{m+1}{2}}|X-U|^{\rho+\alpha-\frac{m+1}{2}}e^{-tr(1-UX^{-1})}dX$$

On evaluating X-integral with the help of result given by Mathai [3]

$$\int_0^1 e^{-tr(XZ)}|X|^{\delta-\frac{m+1}{2}}|I-X|^{\rho-\delta-\frac{m+1}{2}}dX=\frac{\Gamma_m(\delta)\Gamma_m(\rho-\delta)}{\Gamma_m(\rho)}{}_1F_1[\delta;\rho;-Z] \qquad (2.7)$$

For $\mathrm{Re}(\delta)>\frac{m+1}{2},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\rho-\delta)>\frac{m+1}{2}$

We obtain the required result.

Theorem5. If $f(X)\in L_P(0,\infty)\ 1\le P\le 2[or f(X)\in M_P(0,\infty)\ and\ P>2]$ where $\mathrm{Re}(\alpha)>\frac{m+1}{2}$, $\mathrm{Re}(\delta)>-\frac{1}{P},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\alpha-\rho)>\frac{m+1}{2},\frac{1}{p}+\frac{1}{q}=1$ and $|\arg(I-a)|<\pi$ then

$$M\left\{K[^{\alpha}_{\delta,\rho,a};f(X)]\right\}=\frac{\Gamma_m\left(\delta+s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\delta+s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$${}_1F_1\left[\rho+\alpha;\delta+s+\alpha+\rho+\frac{m+1}{2};I\right]M[f(U)] \qquad (2.8)$$

Proof: Using the Mellin transform of

$$K[f(X)] = K\left[{}^{(a_r),(b_r)}_{\sigma,\rho,a}; f(X)\right] =$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{\delta} |X-U|^{\rho-\frac{m+1}{2}} G^{p,q}_{r,s}\left[a(I-XU^{-1})\Big|^{(a_r)}_{(b_r)}\right] f(U)dU \qquad (2.9)$$

We get

$$M\left\{K[^{\alpha}_{\delta,\rho,a}; f(X)]\right\} =$$

$$\int_{X>0} \frac{|X|^{s-\frac{m+1}{2}} |X|^{\delta}}{\Gamma_m(\rho)} \left[\int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-XU^{-1})\Big|^{-}_{(b_1)}\right] f(U)dU\right] dX \qquad (2.10)$$

Changing the order of integration and evaluating X-integral with the help of (2.6), we obtain the required result.

When $M=1, N=0, P=0, Q=1, a=1$ in (2.6) and (2.9) reduces to the following from of operators:

$$R[f(X)] = R\left[{}^{\alpha,\beta}_{\sigma,\rho,1}; f(X)\right] =$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU \qquad (2.11)$$

And

$$K[f(X)] = K\left[{}^{\alpha,\beta}_{\sigma,\rho,1}; f(X)\right] =$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\sigma} |U-X|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU \qquad (2.12)$$

Theorem6. If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2 [or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where $\mathrm{Re}(\alpha) > \frac{m+1}{2}$,

$\mathrm{Re}(\sigma) > -\frac{1}{Q}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\sigma) > \frac{m+1}{2}, \frac{1}{P}+\frac{1}{Q} = 1$ and $|\arg(I-a)| < \pi$ then

$$M\left\{R[^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} = \frac{\Gamma_m(\sigma+\alpha-\beta-s)\Gamma_m(\rho+\beta)}{\Gamma_m(\sigma-s+\alpha+\rho)\Gamma_m(\rho)\Gamma_m(\alpha-\beta)} M[f(U)] \qquad (2.13)$$

Proof: Using the Mellin transform of (2.11), we get

$$M\left\{R[^{\alpha}_{\sigma,\rho,1};f(X)]\right\}=$$

$$\int_{X>0}\frac{|X|^{-\sigma-\rho}|X|^{s-\frac{m+1}{2}}}{\Gamma_m(\rho)}\left[\int_{0<U<X}|U|^{\alpha}|X-U|^{\rho-\frac{m+1}{2}}G^{1,0}_{1,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right]f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\left\{R[^{\alpha}_{\sigma,\rho,1};f(X)]\right\}=\frac{1}{\Gamma_m(\rho)}\int_{U>0}|U|^{\sigma}f(U)dU$$

$$\int G^{1,0}_{1,1}\left[(I-XU^{-1})\Big|^{\alpha}_{\beta}\right]|X|^{s-\sigma-\rho-\frac{m+1}{2}}|X-U|^{\rho-\frac{m+1}{2}}\,dX$$

Using the result given by Mathai [5].

$$G^{1,0}_{1,1}\left[X\Big|^{\alpha}_{\beta}\right]=\frac{1}{\Gamma_m(\alpha-\beta)}|X|^{\beta}|I-U|^{\alpha-\beta-\frac{m+1}{2}} \qquad (2.14)$$

Provided

$0<X<I,\mathrm{Re}(\alpha-\beta)>\frac{m+1}{2}$, We get

$$M\left\{R[^{\alpha,\beta}_{\sigma,\rho,1};f(X)]\right\}=\frac{1}{\Gamma_m(\rho)}\int_{U>0}|U|^{\sigma-\alpha-\beta-\frac{m+1}{2}}f(U)dU$$

$$\frac{1}{\Gamma_m(\alpha-\beta)}\int_{X>U}|X|^{s-\sigma-\alpha}|X-U|^{\beta+\rho-\frac{m+1}{2}}\,dX$$

On evaluating X-integral with the help of the following result

$$\int_0^1|X|^{\delta-\frac{m+1}{2}}|I-X|^{\rho-\frac{m+1}{2}}\,dX=\frac{\Gamma_m(\delta)\Gamma_m(\rho)}{\Gamma_m(\rho+\delta)} \qquad (2.15)$$

For $\mathrm{Re}(\delta)>0,\mathrm{Re}(\rho)>\frac{m+1}{2}$

We arrive at the required result.

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