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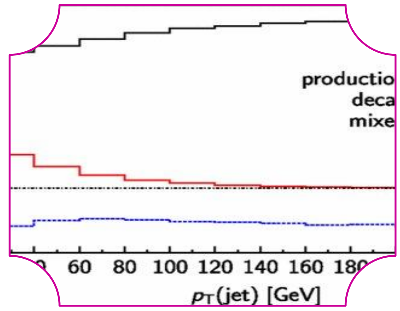
A STUDY ON FRACTIONAL DIFFERINTEGRATIONS IN ASSOCIATION WITH ALEPH- FUNCTION



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ABSTRACT

In the present paper, the author has defined the fractional differintegrations of Aleph $\aleph$-Function in association with different functions of one variable. Corollary and some examples are also given.

KEYWORDS: *fractional differintegrations , integral curves , fractional derivative.*

1. INTRODUCTION

Definitions of the fractional derivatives and integral of the function of single variable:

(i) Goursat's theorem (Cauchy's theorem) for the function of single variable is:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta \qquad (n \in N \cup \{0\}, z \in D) \tag{1.1}$$

Where $f(z)$ is analytic in a domain D, which is surrounded with a piecewise smooth closed Jorden curve γ, in the ζ -plane.

(ii) (Derivative). If $f(z)$ is an analytic (regular) function and it has no branch point inside $C(=\{\underset{-}{C},\underset{+}{C}\})$ and on C, and

$$\underset{-}{{}_C}f_v = \underset{-}{{}_C}f_v(z) = \frac{\Gamma(v+1)}{2\pi i}\int_{\underset{-}{C}}\frac{f(\zeta)}{(\zeta - z)^{v+1}}d\zeta \tag{1.2}$$

$$= \frac{\Gamma(v+1)}{2\pi i}\int_{-\infty}^{(0+)}\eta^{-(v+1)}f(z+\eta)d\eta, \quad (\zeta - z = \eta) \tag{1.3}$$

$$(\zeta \neq z, -\pi \leq \arg(\zeta - z) \leq \pi, v \notin Z^{-})$$

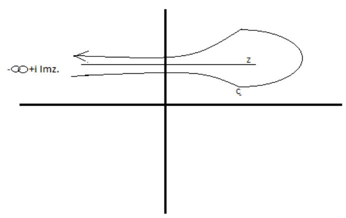
$$\underset{+}{{}_C}f_v = \underset{+}{{}_C}f_v(z) = \frac{\Gamma(v+1)}{2\pi i}\int_{\underset{+}{C}}\frac{f(\zeta)}{(\zeta - z)^{v+1}}d\zeta \tag{1.4}$$

$$= \frac{\Gamma(\nu+1)}{2\pi i} \int_{\infty}^{(0+)} \eta^{-(\nu+1)} f(z+\eta)d\eta, \quad (\zeta - z = \eta) \tag{1.5}$$

$$(\zeta \neq z, -\pi \leq \arg(\zeta - z) \leq \pi, \nu \notin Z^{-})$$

$$f_{-n} = {}_{C}f_{-n} = \lim_{\nu \to -n} {}_{C}f_{\nu} \quad (n \in Z^{+}, C = \{\underset{-}{C}, \underset{+}{C}\}), \tag{1.6}$$

Where $\underset{-}{C}$ and $\underset{+}{C}$ are integral curves as shown in Fig. 1 and Fig. 2 (that is $\underset{-}{C}$ is a curve along the cut joining two points z and $-\infty + i\lim(z)$, and $\underset{+}{C}$ is a curve along the cut joining two points z and $\infty + i\lim(z)$, then $f_{\nu} = {}_{C}f_{\nu}(z) = \{{}_{\underset{-}{C}}f_{\nu}(z), {}_{\underset{+}{C}}f_{\nu}(z)\}(\nu > 0)$ is the fractional derivative of order ν of the function $f(z)$, if f_{ν} exists.



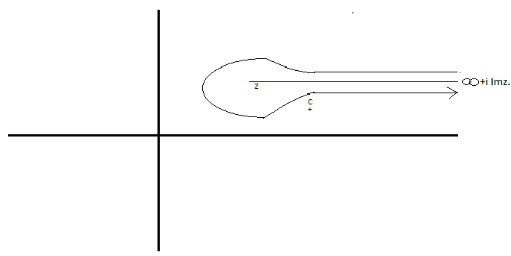


Fig. 1 **Fig. 2**

Definition 2 (Integral). $f_{\nu}(\nu < 0)$ is the fractional integral of order $|\nu|$. That is, the derivative of fractional order $-\nu(\nu > 0)$ is the fractional integral of order $\nu(\nu \in R)$, if f_{ν} exists.

Formal unification of derivative and integral of the function of single variable:

If $f(z)$ is the analytic function and it has no branch point inside C and on $C(C = \{\underset{-}{C}, \underset{+}{C}\})$, and

$$f_{\nu} = {}_{C}f_{\nu}(z) = \{{}_{\underset{-}{C}}f_{\nu}(z), {}_{\underset{+}{C}}f_{\nu}(z)\} \tag{1.7}$$

Then

$$f_{\nu} \text{ is } \begin{cases} \text{derivative for } \nu > 0 \\ \text{original for } \nu = 0 \\ \text{integral for } \nu < 0 \end{cases} \tag{1.8}$$

For $\nu \in R$, and

$$f_{\nu} \text{ is } \begin{cases} \text{derivative for } \mathrm{Re}(\nu) > 0 \\ \text{original for } \nu = 0 \\ \text{integral for } \mathrm{Re}(\nu) < 0 \end{cases} \tag{1.9}$$

For $\nu \in C$, if f_ν exists.

And in case of $\mathrm{Re}(\nu) = 0, f_\nu$ is only formal differintegration regardless of $\mathrm{Im}(\nu) \geq 0$ or $\mathrm{Im}(\nu) \leq 0$. That is , we have no derivative and integral for ν =pure imaginary.

Following results will be used:

(i) ([1];p.16, eq.(1))

$$\left(e^{-az}\right)_\nu = e^{-i\pi\nu} a^\nu e^{-az} \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.10}$$

(ii) ([1];p.18, eq.(6))

$$\left(e^{az}\right)_\nu = a^\nu e^{-az} \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.11}$$

(iii) ([1];p.19, eq.(11))

$$\left(a^{z}\right)_\nu = (\log a)^\nu a^z \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.12}$$

(iv) ([1];p.20, eq.(1))

$$\left(\cosh az\right)_\nu = (-ia)^\nu \cosh(az + i\frac{\pi}{2}\nu) \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.13}$$

(v) ([1];p.20, eq.(2))

$$\left(\sinh az\right)_\nu = (-ia)^\nu \sinh(az + i\frac{\pi}{2}\nu) \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.14}$$

(vi) ([1];p.21, eq.(1))

$$\left(\cos az\right)_\nu = (a)^\nu \cos(az + \frac{\pi}{2}\nu) \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.15}$$

(vii) ([1];p.22, eq.(2))

$$\left(\sin az\right)_\nu = (a)^\nu \sin(az + \frac{\pi}{2}\nu) \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.16}$$

(viii) ([1];p.32, eq.(1))

$$(\log az)_\nu = -e^{-i\pi\nu} \Gamma(\nu) z^{-\nu} \quad \text{for } a \neq 0 (z, \nu \in C) \tag{1.17}$$

The Aleph-function given by Sudland et.al.[6] will be represented and defined in the following manner:

$$\aleph[Z]=\aleph^{m,n}_{p_i,q_i:r}[Z]=\aleph^{m,n}_{p_i,q_i:\tau_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m},[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \chi(s)ds \tag{1.18}$$

where

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji}s)\right\}} \tag{1.19}$$

2. MAIN RESULTS

Theorem 1.

$\left(\aleph(e^{-kz})\right)_v=e^{-i\pi v}(kz^v)^{-1}\aleph(e^{-kz})$ for $k\neq 0(z,v\in C)$

Proof: In case of $|\arg k|<\frac{\pi}{2}$

$$\left(\aleph(e^{-kz})\right)_v=\underset{+}{C}\left(\aleph(e^{-kz})\right)_v=\frac{\Gamma(v+1)}{2\pi i}\int_{\underset{+}{C}}\frac{\aleph(e^{-k\zeta})}{(\zeta-z)^{v+1}}d\zeta$$

$$=\frac{\Gamma(v+1)}{2\pi i}\int_{\underset{+}{C}}\frac{\left\{\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji}s)\right\}}(e^{-k\zeta})^s ds\right\}}{(\zeta-z)^{v+1}}d\zeta$$

$$=\frac{1}{2\pi i}\int_L\theta(s)ds\left\{\frac{\Gamma(v+1)}{2\pi i}\int_{\underset{+}{C}}\frac{e^{-ks\zeta}}{(\zeta-z)^{v+1}}d\zeta\right\}$$

$$=\frac{1}{2\pi i}\int_L\theta(s)e^{-i\pi v}(ks)^v e^{-ksz}ds=e^{-i\pi v}(kz^v)^{-1}\aleph(e^{-kz})$$

Case II. $\frac{\pi}{2}<|\arg z|<\pi$, we have

$$\left(\aleph(e^{-kz})\right)_v=\underset{-}{C}\left(\aleph(e^{-kz})\right)_v=\frac{\Gamma(v+1)}{2\pi i}\int_{\underset{-}{C}}\frac{\aleph(e^{-k\zeta})}{(\zeta-z)^{v+1}}d\zeta$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{-}{C}}\frac{\left\{\frac{1}{2\pi i}\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji}s)\right\}}\int_L (e^{-k\zeta})^s\,ds\right\}}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{1}{2\pi i}\int_L\theta(s)ds\left\{\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{-}{C}}\frac{e^{-ks\zeta}}{(\zeta-z)^{\nu+1}}d\zeta\right\}=e^{-i\pi\nu}(kz^{\nu})^{-1}\aleph(e^{-kz})$$

Case III. $|\arg z|=\frac{\pi}{2}$

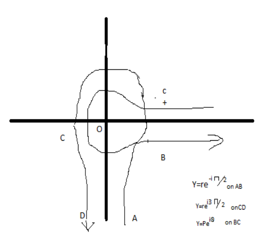
$$\left(\aleph(e^{-kz})\right)_\nu=\underset{+}{C}\left(\aleph(e^{-kz})\right)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{\aleph(e^{-k\zeta})}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{\left\{\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji}s)\right\}}(e^{-k\zeta})^s\,ds\right\}}{(\zeta-z)^{\nu+1}}d\zeta$$

$$=\frac{1}{2\pi i}\int_L\theta(s)ds\left\{\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}}\frac{e^{-ks\zeta}}{(\zeta-z)^{\nu+1}}d\zeta\right\}$$

(put $\zeta-z=\eta, ks\eta=\xi, 0\le|\arg\eta|\le 2\pi$)

$$=\frac{1}{2\pi i}\int_L\theta(s)ds.(ks)^{\nu}e^{-ksz}\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty e^{-i\frac{\pi}{2}}}^{(0+)}\xi^{-(\nu+1)}e^{-\xi}d\xi,\ (\phi=\arg k=-\frac{\pi}{2})\quad (2.1)$$



And

$$\int_{\infty e^{-i\frac{\pi}{2}}}^{(0+)} \xi^{-(\nu+1)} e^{-\xi} d\xi = \left(\int_{AB} + \int_{CD} + \int_{BC} \right) \xi^{-(\nu+1)} e^{-\xi} d\xi$$

$$= \int_{\infty}^{0} \left(re^{-i\frac{\pi}{2}} \right)^{-(\nu+1)} e^{-re^{-i\frac{\pi}{2}}} e^{-i\frac{\pi}{2}} dr + \int_{0}^{\infty} \left(re^{-i\frac{3\pi}{2}} \right)^{-(\nu+1)} e^{-re^{-i\frac{3\pi}{2}}} e^{-i\frac{3\pi}{2}} dr$$

$$+ \lim_{\rho \to 0} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\rho e^{i\theta} \right)^{-(\nu+1)} e^{-(\rho e^{i\theta})} \rho i e^{i\theta} d\theta$$

$$= -2ie^{-i\frac{\pi}{2}\nu} \sin \pi\nu \Gamma(-\nu) e^{-i\frac{\pi}{2}\nu} = -2\pi i e^{-i\pi\nu} \frac{\sin \pi\nu}{\pi} \Gamma(-\nu) = \frac{2\pi i e^{-i\pi\nu}}{\Gamma(\nu+1)}$$

From (2.1), we get

$$= e^{-i\pi\nu} (kz^{\nu})^{-1} \aleph(e^{-kz}) .$$

Theorem 2.

$\left(\aleph(e^{kz}) \right)_{\nu} = \Gamma(\nu+1) (kz^{\nu})^{-1} \aleph(e^{kz})$ for $k \neq 0 (z, \nu \in C)$

Proof: In case of $|\arg k| < \frac{\pi}{2}$

$$\left(\aleph(e^{kz}) \right)_{\nu} = \underset{-}{C} \left(\aleph(e^{kz}) \right)_{\nu} = \frac{\Gamma(\nu+1)}{2\pi i} \int_{\underset{-}{C}} \frac{\aleph(e^{k\zeta})}{(\zeta - z)^{\nu+1}} d\zeta$$

$$= \frac{1}{2\pi i} \int_{L} \theta(s) ds \left\{ \frac{\Gamma(\nu+1)}{2\pi i} \int_{\underset{+}{C}} \frac{e^{ks\zeta}}{(\zeta - z)^{\nu+1}} d\zeta \right\}$$

(put $\zeta - z = \eta, ks\eta = \xi, 0 \leq |\arg \eta| \leq 2\pi$)

$$= \frac{1}{2\pi i} \int_{L} \theta(s) ds . (ks)^{\nu} e^{ksz} \frac{\Gamma(\nu+1)}{2\pi i} \int_{-\infty e^{-i\frac{\pi}{2}}}^{(0+)} \xi^{-(\nu+1)} e^{-\xi} d\xi, \ (\phi = \arg k)$$

$$=\frac{1}{2\pi i}\int_L \theta(s)ds.(ks)^{\nu} e^{ksz}\frac{\Gamma(\nu+1)}{2\pi i}\int_{-\infty}^{(0+)}\xi^{-(\nu+1)}e^{-\xi}d\xi,\ (|\phi|<\frac{\pi}{2})$$

For $|\arg k|<\frac{\pi}{2}$, $\int_{-\infty}^{(0+)}\xi^{-(\nu+1)}e^{-\xi}d\xi=\frac{2\pi i}{\Gamma(\nu+1)}$

We arrive at the required result.

In case of $\frac{\pi}{2}\leq|\arg k|\leq\pi$, we have

$$\left(\aleph(e^{kz})\right)_{\nu}=\underset{+}{C}\left(\aleph(e^{kz})\right)_{\nu}$$

By using similar lines we can prove the result easily.

Corollary:

$\left(\aleph(k^{z})\right)_{\nu}=\Gamma(\nu+1)(\log kz^{\nu})^{-1}\aleph(k^{z})$ for $k\neq 0(z,\nu\in C)$ ()

Proof: We can write as

$$\left(\aleph(k^{z})\right)_{\nu}=\left(\aleph(e^{z\log k})\right)_{\nu}$$

Some Examples:

(i) $\aleph(e^{-5z})_{\frac{1}{2}}=e^{-i\frac{\pi}{2}}(5z^{\frac{1}{2}})^{-1}\aleph(e^{-5z})=-\frac{i}{5\sqrt{z}}\aleph(e^{-5z})$

(ii) $\aleph(e^{-5z})_{-\frac{1}{2}}=-\frac{i\sqrt{z}}{5}\aleph(e^{-5z})$

(iii) $\aleph(e^{3z})_{\frac{1}{2}}=\frac{\sqrt{\pi}}{6\sqrt{z}}\aleph(e^{3z})$

(iv) $\aleph(e^{-3z})_{-\frac{1}{2}}=\frac{\sqrt{\pi z}}{3}\aleph(e^{-3z})$

(v) $\aleph(k^{z})_{\frac{1}{2}}=\frac{\sqrt{\pi}}{2\log 3\sqrt{z}}\aleph(k^{z})=\frac{\sqrt{\pi}}{\log 9z}\aleph(k^{z})$

(vi) $\aleph(k^z)_{-\frac{1}{2}} = \dfrac{\sqrt{\pi}}{2\log(3/\sqrt{z})}\aleph(k^z) = \dfrac{\sqrt{\pi}}{\log(\frac{9}{z})}\aleph(k^z)$

Theorem 3.

$$\aleph(z^k)_\nu = e^{-i\pi\nu} z^{-\nu} \frac{\Gamma(\nu - ks)}{\Gamma(-ks)} \aleph(z^k)$$

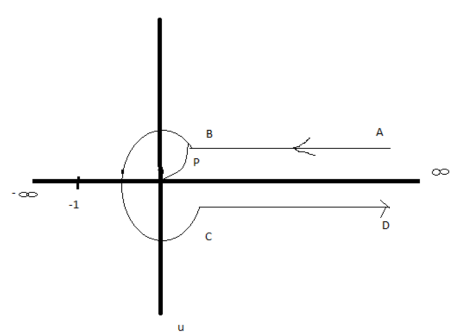
Case I: If $\left|\dfrac{\Gamma(\nu-ks)}{\Gamma(-ks)}\right| < \infty$, we have then

$$\aleph\left(z^k\right)_\nu = \underset{+}{C}\left(\aleph\left(z^k\right)\right)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{C}} \frac{\aleph(\zeta^k)}{(\zeta - z)^{\nu+1}} d\zeta$$

$$= \frac{1}{2\pi i}\int_L \theta(s)ds.\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty e^{i\phi}}^{(0+)} u^{-(\nu+1)}(1+u)^{ks} z^{ks-\nu} du,\ (\phi = \arg z)$$

By putting ($\zeta - z = \eta, \eta = zu$)

$$= \frac{1}{2\pi i}\int_L \theta(s)ds.\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty}^{(0+)} u^{-(\nu+1)}(1+u)^{ks} z^{ks-\nu} du, (\phi < \frac{\pi}{2}) \qquad (2.2)$$



And

$$\int_{\infty}^{(0+)} u^{-(\nu+1)}(1+u)^{ks} du$$

$$= \lim_{\rho \to 0} \left(\int_{AB} + \int_{BC} + \int_{CD} \right) u^{-(\nu+1)} (1+u)^{ks} du \tag{2.3}$$

$$(u = re^{i\theta} on\ AB, u = re^{i2\pi} on\, CD, u = \rho e^{i\theta}\ on\, BC)$$

$$= -\int_0^\infty r^{-(\nu+1)} (1+r)^{ks} dr + e^{-i2\pi\nu} \int_0^\infty r^{-(\nu+1)} (1+r)^{ks} dr + \lim_{\rho \to 0} \rho^{-\nu} \int_0^{2\pi} e^{-i\theta\nu} d\theta$$

$$= (e^{-i2\pi\nu} - 1) \int_0^\infty \int_0^\infty r^{-(\nu+1)} (1+r)^{ks} dr, \ (\mathrm{Re}(\nu) < 0) \tag{2.4}$$

$$e^{-i2\pi\nu} - 1 = -i2e^{-i\pi\nu} \sin \pi\nu = e^{-i\pi\nu} \frac{2\pi i}{\Gamma(\nu+1)\Gamma(-\nu)} \tag{2.5}$$

And $\int_0^\infty r^{-(\nu+1)} (1+r)^{ks} dr = \frac{\Gamma(-\nu)\Gamma(\nu-ks)}{\Gamma(-ks)} (\mathrm{Re}(ks) < \mathrm{Re}(\nu) < 0)$ (2.6)

Applying (2.3), (2.6) into (2.4), we have then

$$\int_\infty^{(0+)} u^{-(\nu+1)} (1+u)^{ks} du = e^{-i\pi\nu} \frac{\Gamma(\nu-ks)}{\Gamma(-ks)} \frac{2\pi i}{\Gamma(\nu+1)} \tag{2.7}$$

Substituting (2.7) into (2.2), we have then

$$= e^{-i\pi\nu} z^{-\nu} \frac{\Gamma(\nu-ks)}{\Gamma(-ks)} \aleph(z^k)$$

For $\mathrm{Re}(ks) < \mathrm{Re}(\nu) < 0, |\arg z| < \frac{\pi}{2}, \left| \frac{\Gamma(\nu-ks)}{\Gamma(-ks)} \right| < \infty$.

Case II: For $\mathrm{Re}(ks) < \mathrm{Re}(\nu) < 0, \frac{\pi}{2} \le |\arg z| \le \pi, \left| \frac{\Gamma(\nu-ks)}{\Gamma(-ks)} \right| < \infty$

In the same way, we have

$$\aleph\left(z^k\right)_\nu = \underline{C}\left(\aleph\left(z^k\right)\right)_\nu$$

$$= \frac{1}{2\pi i} \int_L \theta(s) ds. \frac{\Gamma(\nu+1)}{2\pi i} \int_{-\infty e^{i\phi}}^{(0+)} u^{-(\nu+1)} (1+u)^{ks} z^{ks-\nu} du, \ (\phi = \arg z)$$

$$=\frac{1}{2\pi i}\int_{L}\theta(s)ds.\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty}^{(0+)}u^{-(\nu+1)}(1+u)^{ks}z^{ks-\nu}du,(\frac{\pi}{2}<\phi<\pi)$$

$$=e^{-i\pi\nu}z^{-\nu}\frac{\Gamma(\nu-ks)}{\Gamma(-ks)}\aleph(z^{k})$$

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