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PRELIMINARY STUDY ON METRIC SPACES

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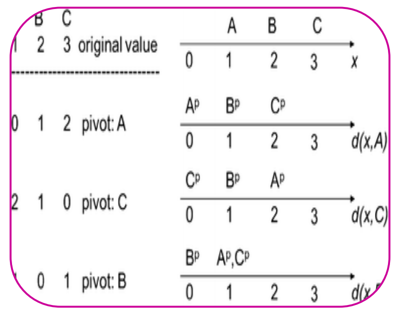
ABSTRACT

Many of the argument in several variable calculus are almost identical to the corresponding argument in one variable calculus, especially argument concerning convergence and continuity. The reason is that the notions of convergence and continuity can be formulated in terms of distance. In more advanced mathematics, we need to find the distance between more complicated objects than numbers and vectors, e.g. between sequences, sets and functions. These new notions of distance leads to new notion of convergence and continuity, and these again lead to new arguments similar to those we have already seen in one and several variable calculus. We can develop a general notion of distance that covers the distances between numbers, vectors, sequences, functions, sets and much more. Within this theory we can formulate and prove results about convergence and continuity once and for all.



KEY WORD: *Metric Space, Norms, Open Spheres, Closed Sphere, Neighbourhoods, Limit Point, Isolated Points, Cauchy Sequence.*

INTRODUCTION :

It is possible to develop a general theory of distance where we can prove the results we need once and for all by the theory of metric spaces.

A metric space is just a set X equipped with a function d of two variables which measures the distance between points: d(x, y) is the distance between two point x and y in X. It turns out that if we put mild and natural conditions on the function d.

PRELIMINARIES

Inequalities

Two real numbers or two algebraic expression related by the symbol<, >, ≤or ≥from an inequality

Examples

1. Triangle inequality

Let ∝, β∈ k then $/\propto+\beta/ \leq /\propto/+/\beta/$

2. Let ∝,β ∈k then $\frac{\propto+\beta}{1+\propto+\beta} \leq \frac{\propto}{1+\propto} + \frac{\beta}{1+\beta}$

3. Holders inequality

Infinite form, let 1<p<∞ &$\frac{1}{p} + \frac{1}{q} = 1. if\ \alpha i, \beta i\ \epsilon k(i = 1,2,3 \dots \dots)$then

$\sum_{i=1}^{n} |\propto i\beta| \leq [\sum_{i=1}^{n} |\propto i| p]1/p[\sum_{i=1}^{n} |\beta i| q]^{1/q}$

Also

$\sum_{i=1}^{n} |\propto i\beta| \leq [\sum_{i=1}^{n} |\propto i|]\max|\beta i|$

Infinite form, let $1<p<\infty$& and q is conjugate to p. if ($\propto_1,\propto_2$ ...) £] p, (β_1, β2...) £] q. i.e.

4. Cauchy-Schwarz inequality

In the finite form, when p=q=2

5. Minkowski's inequality

In the finite form, let $1\leq p<\infty$. If ∝i, βi £ k (i=1, 2... n)

Then

Infinite form, let $1\leq p<\infty$, if ($\propto_1,\propto_2$...), (β_1, β2...) £]^{p}Let $0\leq p\leq 1$, if $\propto_i,\beta_i \in k(i=1,2,\ldots,n)$ then

Metric space

A metric space is a set X that has a notation of the distance d(x, y) between every pair of points x, y ∈X. A metric on a set is function that satisfies the minimal properties we might expect of a distance.

Definition

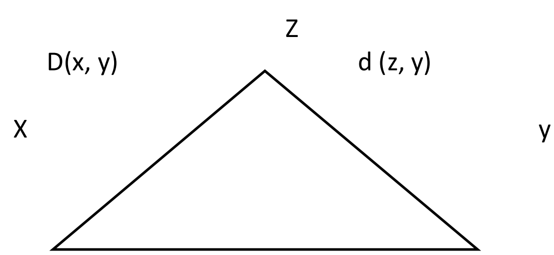
A metric d on a set X is a function d: X x X→R such that for all x, y∈X, if it satisfies the following conditions.

- D(x,y) ≥0, A x,y ∈X
- D(x,y)=0 if and only if x=y,x,y ∈X
- D(X,Y)=d(y,x) A x,y ∈ X(symmetry)
- D(x,y) ≤d(x,z)+ d(z,y) A x,y,y z ∈X (triangle inequality)

The ordered pair (X, d) is called a metric space. A metric space (X, d) is a set X with a metric d defined on X.

Remarks

1. The triangle inequality may be interpreted as that 'the length of one side of a triangle cannot exceed the sum of the lengths of the other two sides'.



2. The triangle inequality can be generalized for any number of additional points $z_1,z_2,\ldots,z_n$ in X

i.e. d(x,y)≤d(x,z1)+d(z2,z2) +...+d(zn,y)

Norms

A normed vector space (x ||.| |) is a vector space X together with a function ||.| |: X→R, called a norm on X, such that for all x, y,∈ X and k ∈R if it satisfies the following properties.

- ⇨ 0≤ ||x| |<∞ and ||x| |=0 if X=0
- ⇨ ||Kx| |=|k| ||x| |
- ⇨ ||x+y| |≤ ||x| |+ ||y| |

The length of x is 0 if x is the 0- vector; multiplying a vector by k multiplies its length by |k|; and the length of the "hypotenuse" x+y is less than or equal to the sum of as lengths of the "sides" x, y. because of this last interpretation, property (3) is referred to as the triangle inequality.

A metric associated with a norm has the additional properties that for all x, y, z ∈ X and K ∈R.

D(x+z, y+z) =d (x'y); d (Kx, K_Y) =|k|d(x, y) which are called translation invariance and homogeneity, respectively. These properties do not even make sense in a general metric space since we cannot add points or multiply them by scalars.

Open spheres

Let (x, d) be a metric space. The open sphere of radius r>0 and centre x∈X is the set $B_r(x)$ ={y ∈X: d(x, y) <r}

An open sphere is always non-empty since it contains its centre at least

Example

i. Consider R with its standards absolute value metric. Then the open ball Br (x) = {y∈R:|X-Y|<R} is the open interval of radius r centred at x.

ii. Let x_0 be any point in the discrete metric space X_D. Then

Sr(x0) = $\begin{cases} \{Xo\}, & 0 < r \leq 1 \\ X, & r > 1 \end{cases}$

iii. In the metric space (R2,d) of the unit sphere centred at the origin is given by, S1 $\left((0,0)\right) = \{(X1, X2)$: $X_1^2+X_2^2<1$}

Closed sphere

Let (x, d) be a metric space. The closed sphere of radius r>0 and centre x ∈ X is the set Br[x] ={y ∈x: d(x, y) ≤r}

Neighbourhoods or open set

Let x be a metric space. A set G ⊂X is open if for every x∈G there exists r>0 such that Br (x) ⊂G.

Theorem

Let (x, d) be a metric space. Then, each open sphere in X is an open set.

Proof

Let $Sr(x_0)$ ={x ∈X: d(x, x_0) <r} be an open sphere in (x, d). Let y_0∈Sr (x_0) be arbitrary but fixed. Then d (xd_0, y_0) <r. write r1=r-d(x_0, y_0).

Clearly r1>0. Consider S_{r1} (y0) ={y ∈x: d(y-y_0) <r_1}

Let y ∈ Sr1 (y_0) be arbitrary. Then d(y, y_o) <r1

Now d(x_0, y) ≤d(x_0, y_0) +d (y_0, y) (by triangle inequality)

<d(x_0, y_0) +r1

=r

⇒ y∈Sr(x_0)

Consequently, Sr1 (y0) ⊂Sr(x0)

This proves that Sr(x0) is a neighbourhood of y0. But y0 ∈ Sr(x0) is arbitrary.

∴ Sr (x0) is a neighbourhood of each of its points. Hence Sr(x0) is an open set.

i. The set $\{1, \frac{1}{2}, \frac{1}{3} \dots \dots\}$ is not an open set

ii. The set of all irrational numbers is not an open set.

iii. In the usual metric space R_u, {x}, x∈ R, is not an open set.

Theorem

Let (x, d) be a metric space and $x \in X$. let N_x be the collection of all neighbourhoods of x. then:

a. $M, N, \in N_x \Rightarrow M \cap N \in N_x$
b. $N \in N_x$ and $M \supset N \Rightarrow M \in N_x$

Proof

a) We have

M, N∈Nx ⇒∃ $r_1, r_2 > 0$ such that $Sr_1(x) \subset M$ and $Sr_2(x) \subset N$

$\Rightarrow S_r(x) \subset M$ and $S_r(x) \subset N$

Where r=min $\{r_1, r_2\}$

$\Rightarrow S_r(x) \subset M \cap N$

⇒M ∩N is a neighbourhood of x

$\Rightarrow M \cap N \in N_x$

b) We have

N ∈Nx ⇒∃ an r >0 such that Sr(x) ⊂N

⇒Sr(x) ⊂M (∴M⊃N)

⇒M ∈Nx

Theorem

Let (x, d) be a metric space. Then

A) Arbitrary union of open sets in x is open.
B) Finite intersection of open sets in x is open

Proof

A) Let $\{G_\propto\}_{\propto\in\Lambda}$ be a family of open sets in x. we shall prove that $\cup_{\propto\in\Lambda} G_\propto$ is open. Since each $G_\propto$ is open, it is union of open spheres for each ∝∈Λ. Then $\cup_{\propto\in\Lambda} G_\propto$ is the union of unions open spheres. Hence union of open sets in x is open.

B) Let {Gi :I=1,2….,n} be the finite family of open sets in x. we shall prove that $\bigcap_{i=1}^{n} Gi \text{ is open}$.

Let x $\in \bigcap_{i=1}^{n} Gi \text{ be arbitrary}$. Then x ∈ Gi, for each I= 1, 2….n

⇒∃ an r1>0 such that Sr1(x) ⊂ Gi, (i=1, 2… n) (∴ each Gi is open)

Let

$$R=\begin{pmatrix} \min ri \\ 1 \le i \le n \end{pmatrix}$$

Then, Sr(x) ⊂Sri(x) ⊂Gi, (i=1, 2… n)

$$\Rightarrow Sr(x) \subset \bigcap_{i=1}^{n} Gi$$

Hence $\bigcap_{i=1}^{n} \quad Gi \text{ is an open set}$

Closed set

Let X be a metric space. A set F ⊂X is closed if Fc ={x ∈X: x ∉F} is open

Example

(I) In the usual metric space R_u, the set A =[1,20 is closed since R-A =]-∞, 1[U] 2, ∞ [IS open.
(II) The set A=R is closed since R-A =Φ is open
(III) The set A=C, the cantor set is closed since by definition the complement of c is open
(IV) In a discrete metric space X_d, a subset Y⊂X is closed.

Theorem

Let (x, d) be a metric space. Then, each closed sphere in x is a closed set.

Proof

Let $S_r[x]$ be a closed sphere in (x, d). It is sufficient to prove that x-$S_r[x]$ is an open set. Let y∈X-$S_r[x]$ be arbitrary. Then y∈X-$S_r[x]$ ∴d(x, y) >r
Let r1=d(x, y)-r. then r1> 0. Let z ∈ Sr(y).
Then d(x, y) <r1
By triangle inequality, we have

D(x, y) ≤d(x, z) + d (z, y)
⇒d(x, z) ≥d(x, y)-d (z, y)
>d(x, y)-r1
=r
⇒z ∈ Sr[x]
⇒z ∈X –Sr[x]

Thus $S_{r1}(y)$ ⊂X-$S_r[x]$ and hence X-$S_r[x]$ is a neighbourhood of y. but y∈X-$S_r[x]$ is arbitrary. ∴X-$S_r[x]$ is a neighbourhood of each of its points. Hence X-$S_r[x]$ is an open set.

Limit point and isolated points

Let (x,d) be a metric space and A⊂X. a point x ∈X is called a limit point of A if each open sphere centred on x contains at least one point of A other than x in other words, if (Sr(x)-{x} ∩A ≠Φ
The set of all limit points of A, denoted by A-1, and is called the derived set of A.

Theorem

Let (x, d) be a metric space and A ⊂ X. then, A is closed if A contains all its limit points. i.e., A^c⊂A.

Proof

Assume that A is closed. We shall prove that Ac ⊂A. let x ∈Ac. Suppose that x ∈A. Then x ∈ A. then x ∈X –A. but A is closed.

∴X-A is open. There exists an r>0 such that Sr(x) ⊂X-A. This shows that the open spheres Sr(x) contain no any points of A which is contradiction. Hence our assumption is wrong. Thus x ∈A.
This proves that A^c⊂A.
Conversely,

Assume that A^c ⊂A. we shall prove that A is closed. Let x ∈X-A.L then x ∈A and also x ∈Ac since A^c ⊂A.

∴We can find an r>0 such that Sr(x) ⊂ X-A. This shows that X-A B is open and hence A is closed
This completes the proof.

Closure of a set

Let (x, d) be a metric space and A ⊂X. The closure of A, denoted by Ā, is the union of A and the set of all its limit point. i.e., Ā=A U

Boundary points

Let (x, d) be a metric space and A ⊂X. A point x ∈X is said to be a boundary point of A if x is neither an interior point of A nor of X-A.

Bounded sets

Let (x, d) be a metric space A set A ⊂X is bounded if there exist x ∈X and 0 ≤R≤∞ such that d(x, y) ≤R A y ∈A

Let A ⊂B_R(X). Then the triangle inequality implies that B_R(X) ⊂B_S(Y)
S=R+d(x, y)
So if it hold for every x ∈ X.
We define the diameter 0 ≤ diam A ≤∞ of a set A ⊂X by
Diam A=sup {d(x, y); x, y ∈A}
Then A is bounded iff diam A <∞

Examples

1. Let x be a set with the discrete metric. Then x is bounded since x=B1 (x) for any x ∈X.
2. Let (X, d) be a metric space. If A=Sr(x) or Sr[x], where x ∈X and r>0, then A is bounded and d (A) ≤2r.
3. Every set in a discrete metric space is bounded.

Convergent sequences

A sequence (x_n) in the metric space x converges to x, written x_n→x as n→∞

Or

$$\left(\begin{matrix} lim \\ n\rightarrow\infty \end{matrix}\right) X_n = x$$

When A ε> 0 ∃ Nn ≥N ⇒X_n∈ bε(x).

In other words, "any neighbourhood of x contains all the sequence from N onwards". This definition generalizes the definition of convergence for real sequence the expression |x_n -x| <ε generalizes to d (x_n, x) <ε which is the same as x_n∈B_ε(x)

Theorem

In a metric space every convergent sequence has a unique limit.

Proof

Let (x, d) be a metric space and {xn} be a convergent sequence in x. let, if possible, the sequence {xn} converge to two points x and y. then, for each ε>0, thee exists positive integers N1 and N2 such that

$$D\,(X_n, x) < \frac{\varepsilon}{2}, \qquad Au \geq N1$$

And

$$D\,(Xn, y) < \frac{\varepsilon}{2}, \qquad Au \geq N2$$

Now,

$$D(x, y) \leq d\,(Xn, x) + d\,(Xn, y)$$

(By triangle inequality)

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

=ε, A n ≥N = max {N1, N2}
⇒ X=Y

This verifies that the limit is unique.

Theorem

In a metric space, every convergent sequence is bounded.

Proof

Let (x, d) be a metric space and {Xn} be a convergent sequence in x such that Xn→x, as n→∞, in (x, d). Then, there exists a positive integer N such that

$$D (Xn, x) <1, A\ n \geq N$$

Write r=max {1; d (Xn, x), 1 ≤n ≤N}

Therefore

$$D (xn, x) \leq r, \quad A\ n \in N$$

So that

$$D (xn, xm) \leq d (xn, x) +d(x, xm)$$
$$\leq 2r, A\ n, m\in N$$

The diameter of the range of the sequence is bounded by 2r.

This proves the result.

Cauchy Sequence

A sequence {xn} in a metric space (x, d) is said to be a Cauchy sequence if it satisfies for each ε>0, there exists a positive integer N such that

$$D (x_m, x_n) <\varepsilon, \quad A\ m, n\geq N$$

Proof

Let xn→x as n →∞. Then, for each ε>0, there exists a positive integer N such that,

$$D (xn, x) <\frac{\varepsilon}{2}, A\ n \geq N$$

Now,

$$D (X_m, Xn) <d (X_m, X) +D (X_n, X)$$

(By triangle inequality)

$$<\frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$=\varepsilon, A\ m, n \geq N$$

Hence the proof.

Theorem

Let $\{x_n\}$ be a convergent sequence in a metric space (x, d) such that xn→x as n→∞. If $\{X_{nk}\}$ is my subsequence of {Xn} then X_{nk}→x as K →∞

Proof

It follows by using the fact that a convergent sequence is a Cauchy sequence and the triangle inequality.

$$D (Xnk, X) \leq d (Xnk, Xn) +d (Xn, X)$$

If subsequence of a sequence in (x, d) is convergent, then the sequence itself need not be convergent; for instance, consider the sequence {Xn} in Ru, where Xn= (-1) n, the subsequence $\{X_{2n}\}$ of $\{X_n\}$ given by

$$X_{2n}=1, A\ n$$

Is such that X_{2n}→1 as n→∞ in Ru and sequence $\{x_n\}$ is not convergent. Even if all the subsequence of a sequence is convergent, the sequence itself need not be so, unless every subsequence of it converges to the same limit.

Hence the proof.

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