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ANALYSIS OF α -M^X / G / 1 QUEUE WITH A WAITING SERVER AND VACATIONS

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ABSTRACT

We consider a single server Non- Markovian queueing system with α - Poisson arrivals and a waiting server and vacations. We call the queueing system as α - M^X /G /1. When α =1 it reduces to the usual classical M^X /G /1 model with batch arrivals. We derive the mean busy period of the system.

KEY WORDS AND PHRASES: - α - *Poisson arrivals, batch arrivals, half Cauchy distribution, Mittag – Leffler function, vacations, waiting server.*

1. INTRODUCTION

The Mittag-Leffler distribution is well suited to queueing systems where arrivals take place at large intervals of time. Such queueing system arise naturally in reliability theory, computer networks , replacement of electronic components (Pillai 1988, 1993) , physics (Weissman et al. 1989 , Weron et al.1996),economics (Mainardi et al. 2000 , Sabatelli et al. 2002., Scalas 2003) , insurance mathematics (Kozubowski 1999) , queueing theory.

Recently many authors have discussed the applications of continuous time random walk (fractal time random walk (FTRW)) in physics and in finance. In physics it is discussed in connection with the relaxation phenomenon of a particle (Weron et al. 1996) and transport properties of disordered system (Weissman et al. 1989). Weron et al. (1996) have shown that the relaxation function defined in one dimensional FTRW model follows the Mittag-Leffler distribution. The relaxation function is a waiting time probability distribution (Weron et al.1995).Weissman et al.(1989) discussed the transport properties of disordered system in terms of FTRW model . Where it is shown that the probability distribution of time interval between successive jumps has the Laplace transform

$$\frac{1}{1+s^{\alpha}},\ s>0,\ 0<\alpha<1,$$

This is the Laplace transform of the Mittag-Leffler distribution.

In finance the basic point is that in financial markets not only the returns but also the waiting time between two consecutive trades can be considered as random variables. A study on waiting times in FOREX exchange and in the nineteenth century Irish stock market was presented by Sabatelli et al. (2002) to fit the Irish data by means of a Mittag-Leffler function. Mainardi et al. (2000) have shown that the marginal waiting time distribution is Mittag-Leffler by fitting the German bund future market.

Apart from these results Enrico-Scalas et al. (2003) fitted the GE time series of October 1999 for three different periods of a day and it is shown that the null hypotheses of exponential distribution is rejected at 5 % significance level judged in terms of Anderson-Darling A^2 values. In addition an empirical analysis performed on 30 DJIA stocks shows that the waiting time survival probability for high frequency data may be closer to non-exponential. Pillai (1988) have shown that the Mittag-Leffler distribution which has thick and long tail provide alternative model where the exponential distribution won't work.

When the inter-arrival time distribution in an evolutionary process is Mittag- Leffler with Laplace transform $\frac{\lambda^\alpha}{\lambda^\alpha + s^\alpha}$ $(\lambda > 0, 0 < \alpha < 1, s > 0)$, the corresponding arrival process {N(t), t$\geq$0} is α - Poisson and has the probability density function (Anil 2001, Saji Kumar 2003)

$$P_n(t) = P(N(t) = n)$$
$$= \sum_{k=0}^{\infty} (-1)^k \binom{k+n}{n} \frac{(\lambda t)^{\alpha(k+n)}}{\Gamma(\alpha(k+n)+1)}, \qquad n=0,1,2, \quad . \quad . \quad . \qquad (1.1).$$

When α =1, (1.1) is the probability density function of the usual Poisson distribution with parameter λt.

For h > 0

$\frac{P_n(t+h) - P_n(t)}{h} = \frac{\alpha}{t}\left[nP_n(t) - (n+1)P_{n+1}(t)\right] + o(h)$ where $o(h)$ are terms containing h and higher powers of h

Hence $$\frac{t}{\alpha} P_n'(t) = -(n+1)P_{n+1}(t) + nP_n(t) \qquad (1.2)$$

when $\alpha = 1$, (1.2) is same as (2.7) in chapter XVII, in Feller (1968)

In the analysis to follow we denote the Laplace transform of a function by the same function with an asterisk and argument s, the probability generating function (PGF) with a caret and argument z, unless otherwise specified.

REMARK 1.1

A random variable X is said to have an α - Poisson distribution with parameter λ if it has the probability density mass function

$$P(x) = \sum_{k=0}^{\infty} (-1)^k \binom{k+x}{x} \frac{\lambda^{\alpha(k+x)}}{\Gamma(\alpha(k+x)+1)}, \quad x = 0,1,2,...$$

$P(x)$ has the PGF, the Mittag-Leffler function

$$\hat{P}(z) = E_\alpha(-\lambda^\alpha(1-z)), 0 < z < 1 \qquad (1.3)$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} z^k / \Gamma(k\alpha + 1)$

$P(x)$ is the discrete version of the α - inverted stable distribution, the first passage time distribution of the α -stable Levy motion (Saji Kumar 2002, 2003), the corresponding density function is

$$g(y) = \frac{1}{\pi\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(k\alpha+1)}{k!}\lambda^{-k\alpha}y^{k-1}\sin(\pi k\alpha), y > 0$$

$g(y)$ has the Laplace transform (Feller 1966), the Mittag-Leffler function

$$g^*(s) = E_\alpha(-\lambda^\alpha s)$$

By Feller (1966 pp 359-360) the probability distribution of the number of renewal epochs N_t within (0, t] ,(t > 0) asymptotically follows g(y), when the waiting time density between successive renewals has infinite expectation. Since α -inverted stable distribution is the continuous version of the α -Poisson distribution ,we can conclude that in both asymptotic and non -asymptotic cases the inter-arrival time of renewals follows the Mittag-Leffler distribution when the waiting time density between successive renewals has infinite expectation.

In sequel we frequently use the half Cauchy distribution with a special reparametrization, which is the ratio of two one sided identically distributed positive stable random variables having the Laplace transform $\exp(-s^\alpha)$. The respective probability density function and distribution function are (Saji Kumar 2002)

$$h(y) = \frac{\sigma^\alpha}{\pi}\frac{y^{\alpha-1}\sin\pi\alpha}{y^{2\alpha}+2\sigma^\alpha y^\alpha\cos\pi\alpha+\sigma^{2\alpha}}; y > 0, \sigma > 0$$

and $$H(y) = \frac{1}{\pi\alpha}\tan^{-1}\left(\frac{\sin\pi\alpha}{\left(\frac{\sigma}{y}\right)^\alpha+\cos\pi\alpha}\right) \quad (1.4)$$

$h(y)$ has the Laplace transform, the Mittag-Leffler function (Saji Kumar 2002)

$$H^*(s) = \sum_{k=0}^{\infty}(-1)^k\frac{(s\sigma)^{k\alpha}}{\Gamma(k\alpha+1)}$$
$$= E_\alpha(-(s\sigma)^\alpha) \quad (1.5)$$

We consider a single server Non-Markovian queuing system in which the random batches of customers arrive according to an α -Poisson law with mean $\lambda^\alpha / \Gamma(\alpha+1)$. These models arise naturally in a production line, in sending message through the network, replacement of electronic components, communication problems and other fields (Pillai 1988, 1993 and references there in) and thus α -Poisson distribution have a wide range of applications. When messages are sent through a network, some may be returned and must be sent again. In a telephone centre, some calls may not be completed due to signal failure and must be reestablished. In an assembled unit of electronic components, some items must be replaced again. In such situations as an inter-arrival time distribution, the Mittag-Leffler probability law better fits the data than the exponential distribution (Pillai 1988, 1993).

Optimal operating policies for M/G/1 system in which the server may be turned on and off have been studied over a period of more than three decades and an active area of research. Earlier authors include Yadin et al. (1967), Sobel (1969), Bell (1971), Yechiali (2004), Salehi-Rad et al. (2004). Our purpose is to extent the results and gets new ideas for the queuing system with α -Poisson arrivals and Mittag-Leffler inter-arrival time distribution.

The paper is organized as follows. The section 2 is devoted to queue with a server vacations. In section 3 we find the mean of the busy period.

2. QUEUE WITH SERVER VACATIONS

Queueing system with server vacations reflects many real life situations (Yechiali 2004). The queuing procedure is: Batches of customers are admitted to service according to their order of arrival, within a batch the service pattern is governed by first come first serve (FCFS) rule. The server takes a random vacation U if the queue is empty and returns to the system at the end of U and starts service until the queue becomes empty and goes for another vacation. But if the system is empty during the termination of a random vacation U, the server activates a random timer and waits. If a batch arrives before T expires, the server starts his service until the queue becomes free before taking another vacation U. If no batches arrive during T (that is the inter - arrival time is greater than T) the server takes another random vacation U. The inter – arrival times, batch sizes, service times, vacation length are assumed to be independent.

Each batch size X, has probability density function $P(X = m) = f_m (m = 1,2,3,.....)$ with probability generating function (PGF) $\hat{X}(z) = E(Z^x) = \sum_{m=1}^{\infty} f_m z^m$ We let f=f$^{(1)}$=E(X), f$^{(2)}$ =E(X(X-1)), where the k-th factorial moment $f^{(k)} = d^k \hat{X}(z) / dz^k |_{z=1}$ .Customers are served one at a time by a single server and service times, M , of individual customers are i.i.d. random variables with LST M*(s),mean E (M) = m. The traffic load is denoted by ρ = $\dfrac{f\lambda^{\alpha} E(M^{\alpha})}{\Gamma(\alpha+1)}$.

Here we derive the mean of the number of customers in the system at service completion epochs. Let L_n denotes the number of customers left behind by the nth departing customer. Then the law of motion of the systems state L at departure epoch is given as follows: (Yechiali 2004)

If $L_n > 0$ $\quad L_{n+1} = Ln-1 + \sum_{j=1}^{N(M)} X_j,$ (2.1)

where N(M) represents the number of batches that arrives during the service time M_{n+1} of the (n+1) – th customer, and X_1, X_2, … are i.i.d copies of X. Provided the arriving customers are served immediately from the beginning of the arrival intervals. .

If $L_n = 0$

$$L_{n+1} = \begin{cases} \sum_{j=1}^{\xi} X_j - 1 + \sum_{i=1}^{N(M)} X_i \ , \ \text{w.p} \ \dfrac{1 - V^*(\lambda)}{1 - V^*(\lambda) W^*(\lambda)} = \beta \\ X - 1 + \sum_{i=1}^{N(M)} X_i , \ \text{w.p} \ \dfrac{V^*(\lambda)\left(1 - W^*(\lambda)\right)}{1 - V^*(\lambda)\, W^*(\lambda)} = 1 - \beta \end{cases} \quad (2.2)$$

where ξ = N(U) / N(U) ≥ 1

P(No arrivals in T) = $\int_o^\infty \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda t)^{n\alpha}}{\Gamma(n\alpha+1)}$ dT(t)

P(No arrivals in T) = $\int_o^\infty H^*(\lambda t)\, dT(t)$ (2.3)

By Parseval relation (Feller 1966, XIII.11.3), (2.3) is the Laplace transform of the product W = H T.

Hence P (No arrivals in T) = $W^*(\lambda)$ (2.4)

where $W^*(\lambda)$ is the Laplace transform of W
Similarly

P (No arrivals in U) = $\int_o^\infty H^*(\lambda t)\, dU(t)$

= $V^*(\lambda)$, (2.5) = P(N(U) = 0)

where V = H U and $V^*(\lambda)$ is the Laplace transform of V. The explanation of (2.2) is similar to that of Boxma et al. (2002) and Yechiali (2004).

Sajikumar and Pillai (2006) derived the probability generating function of the queue size at a service completion epoch of this queueing system.

The generating function

$$P_o = \frac{[\Gamma(\alpha+1) - f\lambda^\alpha E(M^\alpha)][1 - V^*(\lambda)W^*(\lambda)]}{f[\lambda^\alpha E(U^\alpha) + V^*(\lambda)(1 - W^*(\lambda))\Gamma(\alpha+1)]} \quad (2.6)$$

Finally

$$\hat{L}(z) = \frac{[\Gamma(\alpha+1) - f\lambda^\alpha E(M^\alpha)]}{f[\lambda^\alpha E(U^\alpha) + V^*(\lambda)(1 - W^*(\lambda))\Gamma(\alpha+1)]} \frac{B^*(\delta)}{B^*(\delta) - z} .[V^*(\lambda)(1-W^*(\lambda))(1-\hat{X}(z)) - V^*(\delta) + 1] \quad (2.7)$$

When α = 1, equations (2,6) and (2.7) reduces to equation (3.10) and (3.11) in Yechiali (2004) . When X $\equiv$ 1, and α = 1 equations (2.7) reduces to equation (3.5) in Boxma et al. (2002) .

When T = ∞ (i.e. $W^*(\lambda)$ = 0, using (2.4)), the timer model reduces to the system with single vacations (SV). Thus (2.7) reduces to

$$\hat{L}(z)_{SV} = \frac{[\Gamma(\alpha+1) - f\lambda^\alpha E(M^\alpha)]}{f[\lambda^\alpha E(U^\alpha) + V^*(\lambda)\Gamma(\alpha+1)]} \frac{B^*(\delta)}{B^*(\delta) - z} [V^*(\lambda)(1-\hat{X}(z)) - V^*(\delta) + 1] \quad (2.8)$$

If T = 0 (i.e. $W^*(\lambda)$ = 1), we obtain queue with Multiple vacations (M.V). Equation (2.7) becomes

$$\hat{L}(z)_{MV} = \frac{[\Gamma(\alpha+1) - f\lambda^\alpha E(M^\alpha)]}{f\lambda^\alpha E(U^\alpha)} \frac{B^*(\delta)}{B^*(\delta) - z} .[1 - V^*(\delta)]$$

$$. = \frac{\hat{L}(z)\left(1-V^*(\delta)\right)}{\frac{\delta^\alpha}{\Gamma(\alpha+1)}E\left(U^\alpha\right)} \tag{2.9}$$

If we denote R_u, the residual part of the vacation time U, then $R_u^*(\delta) = \left(1-V^*(\delta)\right) \Big/ \frac{\delta^\alpha E\left(U^\alpha\right)}{\Gamma(\alpha+1)}$ represents the PGF of the total number of customers arriving during R_U. Equation (2.9) reveals the decomposition properly that the number of customers at a service completion instant is the sum of the number of customers arriving in a queue without server and vacations and the number of customers arriving during the remaining time of a vacation, R_U.

When U=0, equation (2.9) reduces to the PGF of a non-Markovian queue with α -Poisson batch arrivals of departure epochs.

$$\hat{L}(z) = \frac{[\Gamma(\alpha+1) - f\lambda^\alpha E\left(M^\alpha\right)]}{f} \; \frac{B^*(\delta)}{B^*(\delta)-z} .[\, 1- X^{\wedge}(z)\,] \tag{2.10}$$

When α = 1, equation (2.1o) coincides with equation (3.18) in Shomrony et al. (2001) and with equation (2.10) in Cochen (1982) . When X ≡ 1, we have $\hat{X}(z)$ = z, f = 1 and for α = 1 (2.10) reduces to the Khintchine – Pollazcek formula for the classical M|G|1 queue (see Levy et al. 1975, Takagi, 1991)

$$\hat{L}(z) = \frac{(1-\rho)\,(1-z)M^*(\lambda(1-z))}{M^*(\lambda(1-z))-z}$$

3. BUSY PERIOD

Here we derive the mean of the number of customers.

LEMMA 3.1

For α -Poisson arrivals the LST of the busy period satisfies the functional equation

$$\beta(s) = \int_0^\infty E_\alpha\left(-(\lambda t)^\alpha (1-\beta(s))\right)e^{-st}\, dB(t)$$

where B and β denote the distribution function and Laplace transform of the busy period.

PROOF

Let the first customer depart at epoch t and N be the number of customers joined the queue during his service time. N is an α -Poisson variable with parameter λ . The total service time required by the N customers

$$S_N = X_1+X_2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots + X_N,$$

where each X_j have the Laplace transform β . The total duration of the service times has the same distribution as the busy period $t + S_N$

By Feller (1966, pp 448-449) and using (1.3)

$$\beta(s)=\int_0^{\infty} E_{\alpha}(-(\lambda t)^{\alpha}(1-\beta(s)))e^{-st}dB(t) \qquad (3.1)$$

when α =1, (3.1) is same as (4.1) in Feller (1966, pp 448-449)

A busy period starts either with $\xi = N(U)|_{N(U)\geq 1}$ batches that arrived during a vacation U, or with a batch of size X arriving within a timer's duration T. As it is stated in (2.2) β and 1-β are their respectively probability of occurrence. θ_x denotes the duration of a busy period starts with the arrival of a batch of size X.

$$Y=\sum_{i=1}^{X} M_i, \qquad (3.2)$$

is the total service time, M_i are i.i.d. copies of M.

By Feller (1968) the LST of Y is, $Y^*(s) = \hat{X}(M^*(s))$

$E(Y) = f\,m$ and $E(Y^2) = f^{(2)} m^2 + f\, m^{(2)}$

The LST of θ_X,

$$\theta_X^*(s)=\int_0^{\infty} E_{\alpha}\left(-(\lambda t)^{\alpha}(1-\theta_X^*(s))\right)e^{-st}dY(t) \qquad (3.3)$$

$$E(\theta_X) = \frac{E(Y)}{1-\frac{\lambda^{\alpha}}{\Gamma(\alpha+1)}E(Y^{\alpha})}, \qquad (3.4)$$

$$E(\theta_X^2)=\frac{E(Y^2)+2\frac{\lambda^{\alpha}}{\Gamma(\alpha+1)}E(Y^{\alpha+1})E(\theta_X)+2\frac{\lambda^{2\alpha}}{\Gamma(2\alpha+1)}E(Y^{2\alpha})\left(E(\theta_X)\right)^2}{1-\frac{\lambda^{\alpha}}{\Gamma(\alpha+1)}E(Y^{\alpha})} \qquad (3.5)$$

Where $E(Y^{\alpha})$, $E(Y^{\alpha+1})$ and $E(Y^{2\alpha})$ can be calculated by the method of fractional calculus for a known $Y^*(s)$ (see Wolfe (1975))

Let $\theta_{\xi} = \sum_{j=1}^{\xi}(\theta_X)_j$, represents the duration of busy period starts with ξ batchs,where $(\theta_X)_j$ are i.i.d. copies of θ_X.Thus

$$\text{The LST } \theta_{\xi}^*(s) = \frac{V^*\left(\lambda(1-\theta_X^*(s))^{1/\alpha}\right)-V^*(\lambda)}{1-V^*(\lambda)} \qquad (3.6)$$

$$E(\theta_\xi) = \frac{\lambda^\alpha E(U^\alpha)E(\theta_X)}{\Gamma(\alpha+1)\left(1-V^*(\lambda)\right)}$$

$$E(\theta_\zeta^2) = \frac{1}{1-V^*(\lambda)}\left[2\frac{\lambda^{2\alpha}E(U^{2\alpha})}{\Gamma(2\alpha+1)}\left(E(\theta_X)\right)^2 + \frac{\lambda^\alpha E(U^\alpha)}{\Gamma(\alpha+1)}E(\theta_X^2)\right] \quad (3.7)$$

The LST of the busy period θ is given by

$$\theta^*(s) = (1-\beta)\theta_X^*(s) + \beta\theta_\zeta^*(s)$$

$$= \frac{1}{1-V^*(\lambda)W^*(\lambda)}\left\{ \left[V^*(\lambda)\left(1-W^*(\lambda)\right)\right]\int_0^\infty E_\alpha\left(-(\lambda t)^\alpha(1-\theta_X^*(s))\right)e^{-st}dY(t) + V^*\left[\lambda(1-\theta_X^*(s))^{1/\alpha}\right] - V^*(\lambda) \right\} \quad (3.8)$$

Now, $$E(\theta) = \frac{E(\theta_X)}{1-V^*(\lambda)W^*(\lambda)}\left[V^*(\lambda)\left(1-W^*(\lambda)\right) + \frac{\lambda^\alpha E(U^\alpha)}{\Gamma(\alpha+1)}\right] \quad (3.9)$$

$$E(\theta^2) = (1-\beta)E(\theta_X^2) + \beta E(\theta_\zeta^2), \quad (3.10)$$

For the multiple vacation, W*(λ) =1 , equation (4.8) reduces to

$$\theta^*(s)_{\alpha-M^X/G/1+MV} = \frac{1}{1-V^*(\lambda)W^*(\lambda)}\left\{ V^*\left[\lambda(1-\theta_X^*(s))^{1/\alpha}\right] - V^*(\lambda) \right\}$$

$$E(\theta\ (s)_{\alpha-M^X/G/1+MV}) = \frac{E(\theta_X)}{1-V^*(\lambda)}\left[\frac{\lambda^\alpha E(U^\alpha)}{\Gamma(\alpha+1)}\right]$$

$$= E(\theta_\xi)$$

For the single vacation case, T=∞ and W*(λ) =0 $\theta^*(s)_{\alpha-M^X/G/1+SV} = V^*(\lambda)$

$$\int_0^\infty E_\alpha\left(-(\lambda t)^\alpha(1-\theta_X^*(s))\right)e^{-st}dY(t) + V^*\left(\lambda\left(1-\theta_X^*(s)\right)^{1/\alpha}\right)$$

and $E(\theta\ (s)_{\alpha-M^X/G/1+SV}) = E(\theta_X)\left[V^*(\lambda) + \frac{\lambda^\alpha E(U^\alpha)}{\Gamma(\alpha+1)}\right]$

When α =1, all results stated in this of work reduces to the corresponding results in the M^X /G /1 queue with vacations and timer. See Yechiali (2004).

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