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FREDHOLM INTEGRAL EQUATIONS: A SOLUTION PROCEDURE USING FUZZY APPROACH

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Abstract:

The current research attempts to offer a new method for solving fuzzy linear Fredholm integral equations. This method converts the given fuzzy system into a linear system in crisp case by using the approximation method and separable kernel method respectively. Now the solution of this system yields the unknown coefficients of the solution functions. The proposed method is illustrated by an example.

KEYWORDS:

Fuzzy Integral Equation, Fredholm Integral Equations, Approximate Solution, Basic Function, Fuzzy Transforms.

In a real field Fuzzy set theory is a powerful tool for modelling uncertainty and for processing vague or subjective information in mathematical models. The topics of fuzzy differential equations (FDE) and fuzzy integral equations (FIE) attracted growing interest of researchers in recent time. The concept of fuzzy sets which was originally introduced by Zadeh [10] led to the definition of the fuzzy number and its implementation in fuzzy control [6] and approximate reasoning problems [11, 12].

This paper contains the basic material to be used in solving FIE under uncertain condition.

INTRODUCTION:

In the recent years, various powerful mathematical methods based on quadrature formulas [1], Adomian decomposition [1], Legendre wavelets, hybrid Legendre and block-pulse functions [4], iterative interpolation [3] and successive approximation method [2] have been proposed to obtain approximate-analytical solutions for linear fuzzy Fredholm integral equations of the second kind. Recently, Bernstein polynomials have been used to solve linear and nonlinear differential equation as well as some classes of integral equations in the crisp case. These polynomials defined on an interval form a complete basis over the interval. Each of these polynomials is positive and their sum is unity. Using these two properties, we construct the proposed method.

The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [13, 14], Nahmias [15], Dubois and Prade [16, 17], and Ralescu [18].

The fuzzy mapping function was introduced by Chang and Zadeh [7]. Later, Dubois and Prade[8] presented an elementary fuzzy calculus based on the extension principle [10]. Puri and Ralescu [19] suggested two definitions for fuzzy derivative of fuzzy functions. The first method was based on the H-difference notation and wasfurther investigated by Kaleva [9]. The second method was derived from the embedding technique and was followed by Goetchel and Voxman [20] who gave it a more applicable representation. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [8]. Alternative approaches were later suggested by Goetschel and Voxman [20], Kaleva [9], Matloka [21], Nanda [22], and others.

A new method is proposed for approximate the solution of a fuzzy linear Fredholm integral equations system. This method converts the given fuzzy system that supposedly has a unique fuzzy solution, into crisp linear system. This paper presents some preliminary basic definitions and published results by research scholars in this area. In the last section, numerical example is illustrated.

It is well known form of the Fredholm integral equation of the second kind is:

$$F(t) = f(t) + \lambda (ku)(t), \qquad \dots (1)$$

where

$$(ku)(t) = \int_a^b k(s,t)F(s)ds, \qquad a \le t \le b.$$

In (1), $k(s,t)$ is an arbitrary kernel function over the square $a \le s,\ t \le b$ and $f(t)$ is a function of $t : a \le t \le b$.

In addition, if $f(t)$ be a crisp function, then the solution of the above equation is crisp as well. Also if $f(t)$ be a fuzzy function, we have Fredholm's fuzzy integral equation of the second kind which may only process fuzzy solutions. Sufficient conditions for the existence and uniqueness of the solution of the second kind equation, where $f(t)$ is a fuzzy function, are given in [5].

FUZZY INTEGRAL EQUATION:

The Fredholm integral equation of the second kind is

$$F(t) = f(t) + \lambda \int_a^b K(s,t)F(s)ds, \qquad \dots (2)$$

where $\lambda > 0$, K(s, t) is an arbitrary given kernel function over the square $a \le t, s \le b$ and $f(t)$ is given function of $t \in [a,b]$. If $f(t)$ is a crisp function then the solution of above equation is crisp as well. However, if f(t) is a fuzzy function this equation may only possess fuzzy solution. Sufficient conditions for the existence equation of the second kind, where f(t) is a fuzzy function are given in [1]. Now, we introduce parametric form of a FFIE-2 (Fuzzy Fredholm Integral Equation of second kind).

Definition: Let $(\underline{f}(t,r), \overline{f}(t,r))$ and $(\underline{u}(t,r), \overline{u}(t,r))$, $0 \le r \le 1$ and $t \in [a,b]$ are parametric form of f(t) and u(t), respectively then, parametric form of FFIE-2 is as follows:

$$\underline{u}(t,r) = \underline{f}(t,r) + \lambda \int_a^b v_1(s,t,\underline{u}(s,r),\overline{u}(s,r))ds,$$

$$\overline{u}(t,r) = \overline{f}(t,r) + \lambda \int_a^b v_2(s,t,\underline{u}(s,r),\overline{u}(s,r))ds. \qquad \text{… (3)}$$

Where

$$v_1(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} K(s,t)\underline{u}(s,r), K(s,r) \ge 0, \\ K(s,t)\overline{u}(s,r), K(s,r) < 0. \end{cases}$$

$$v_2(s,t,\underline{u}(s,r),\overline{u}(s,r)) = \begin{cases} K(s,t)\overline{u}(s,r), K(s,r) \ge 0, \\ K(s,t)\underline{u}(s,r), K(s,r) < 0. \end{cases}$$

for each $0 \le r \le 1$ and $a \le t \le b$.

We can see that (3) is a system of linear Fredholm integral equations in crisp case for each $0 \le r \le 1$ and $a \le t \le b$.d

LINEAR FREDHOLM INTEGRAL EQUATION USING FUZZY TECHNIQUE:

In this section we present a new method for solving the linear fuzzy Fredholm integral equation with degenerate kernel. The proposed approach will be illustrated in terms of the following equation:

$$F(t) = f(t) + \lambda \int_a^b k(s,t)F(s)ds,$$

with $\lambda > 0$. It is assumed that kernel $k(s, t)$ is degenerate, that is,

$$k(s,t) = \sum_{i=1}^{n} a_i(s)b_i(t),$$

Where $a_i(s)$ and $b_i(t)$, $i = 1, 2, ..., n$, are linearly independent functions. In equation (2), if f is a crisp function then the solution is crisp as well, and in the case that f is a fuzzy function, we have Fredholm fuzzy integral equation of the second kind which may only process fuzzy solutions.

Now we introduce parametric form of the fuzzy integral Eqs. (2) as we know that. Let ($f(t, r)$, $f(t, r)$) and ($F(t, r)$, $F(t, r)$) ($0 \le r \le 1$, $a \le t \le b$) are parametric form of $f(t)$ and $F(t)$ respectively, then the parametric form of the fuzzy integral equation (2) is as follows:

$$\underline{F}(t,r) = \underline{f}(t,r) + \lambda \int_a^b \underline{k(s,t)F(s,r)}ds, \qquad \text{… (4)}$$

$$\overline{F}(t,r) = \overline{f}(t,r) + \lambda \int_a^b \overline{k(s,t)F(s,r)}ds, \qquad \dots (5)$$

where,

$$\underline{k(s,t)F(s,r)} = \begin{cases} k(s,t)\underline{F}(s,r), & k(s,t) \geq 0 \\ k(s,t)\overline{F}(s,r), & k(s,t) < 0 \end{cases}$$

$$\overline{k(s,t)F(s,r)} = \begin{cases} k(s,t)\overline{F}(s,r), & k(s,t) \geq 0 \\ k(s,t)\underline{F}(s,r), & k(s,r) < 0 \end{cases}$$

By the above assumptions the equations (4) and (5) will become in the following forms respectively.

$$\underline{F}(t,r) = \underline{f}(t,r) + \lambda \sum_{i=1}^{n} \int_a^b \underline{a_i(s)b_i(t)F(s,r)}ds, \qquad \dots (6)$$

$$\overline{F}(t,r) = \overline{f}(t,r) + \lambda \sum_{i=1}^{n} \int_a^t \overline{a_i(s)b_i(t)F(s,r)}ds \qquad \dots (7)$$

If we denote by A_i, i= 1, 2,...,n the set of union on subintervals of [a, b] that $a_i(s)$ is nonnegative on these subintervals and by B_i, i= 1, 2, ..., n the set of union on subintervals of [a, b] that $a_i(s)$ is negative on these subintervals. It is clear that $A_i \cup B_i = [a,b]$, i= 1, 2, ..., n.

Without loosing generality, we suppose that $b_i(t)$ is nonnegative for fixed t and $1 \leq i \leq n$. By the above assumptions we can rewrite equations (6) and (7) respectively as follows:

$$\underline{F}(t,r) = \underline{f}(t,r) + \lambda \sum_{i=1}^{n} \left(\sum_{I_i \in A_i} \int_{I_i} a_i(s)b_i(t)\underline{F}(s,r)ds + \sum_{J_i \in B_i} \int_{J_i} a_i(s)b_i(t)\overline{F}(s,r)ds \right), \quad \dots (8)$$

$$\overline{F}(t,r) = \overline{f}(t,r) + \lambda \sum_{i=1}^{n} \left(\sum_{I_i \in A_i} \int_{I_i} a_i(s)b_i(t)\overline{F}(s,r)ds + \sum_{J_i \in B_i} \int_{J_i} a_i(s)b_i(t)\underline{F}(s,r)ds \right), \quad \dots (9)$$

The above equations. We can affirm that,

$$S_F(t,r) = S_f(t,r) + \lambda \sum_{i=1}^{n} \int_a^b a_i(s)b_i(t)S_F(s,r)ds, \qquad \dots (10)$$

$$D_F(t,r) = D_f(t,r) + \lambda \sum_{i=1}^{n} \int_a^b |a_i(s)||b_i(t)|D_F(s,r)ds, \qquad \dots (11)$$

Note that in the negative case of $b_i(t)$ for some i that same results as the above equations are obtained. It emerges that the technique of solving equations (8) and (9) are

$$c_i = \int_a^b a_i(s) S_F(s,r) ds,$$

$$d_i = \int_a^b |a_i(s)| D_F(s,r) ds,$$

By substituting c_i and d_i in (10) and (11) respectively, we obtain

$$S_F(t,r) = S_F(t,r) + \lambda \sum_{i=i}^{n} c_i b_i(t), \qquad \text{… (12)}$$

$$D_F(t,r) = D_f(t,r) + \lambda \sum_{i=i}^{n} d_i |b_i(t)|, \qquad \text{… (13)}$$

By substituting (13) into *ci* we obtain

$$\begin{aligned}\sum_{i=1}^{n} c_i b_i(t) &= \sum_{i=1}^{n} b_i(t) \int_a^b a_i(s) S_F(s,r) ds \\ &= \sum_{i=1}^{n} b_i(t) \int_a^b a_i(s) \left(S_f(s,r) + \lambda \sum_{k=1}^{n} c_k b_k(s) \right) ds\end{aligned}$$

therefore we get

$$\sum_{i=1}^{n} b_i(t) \left\{ c_i - \int_a^b a_i(s) \left(S_f(s,r) + \lambda \sum_{k=1}^{n} c_k b_k(s) \right) ds \right\} = 0$$

in a similar manner we get

$$\sum_{i=1}^{n} |b_i(t)| \left\{ d_i - \int_a^b |a_i(s)| \left(D_f(s,r) + \lambda \sum_{k=1}^{n} d_k |b_k(s)| \right) ds \right\} = 0$$

Since the functions $b_i(t)$ and consequently $|b_i(t)|$, $i = 1, 2, ..., n$ are linearly independent, therefore,

$$c_i - \int_a^b a_i(s) \left(S_f(s,r) + \lambda \sum_{k=1}^{n} c_k b_k(s) \right) ds = 0 \qquad \text{… (14)}$$

$$d_i - \int_a^b |a_i(s)| \left(D_f(s,r) + \lambda \sum_{k=1}^{n} d_k |b_k(s)| \right) ds = 0 \qquad \dots (15)$$

For these computations in the sense of the unknowns, we simplify the problems by using the following notations

$$f_i^{(1)} = \int_a^b a_i(s) S_f(s,r) ds, \qquad a_{ik}^{(1)} = \int_a^b a_i(s) b_k(s) ds,$$

$$f_i^{(2)} = \int_a^b |a_i(s)| D_f(s,r) ds, \qquad a_{ik}^{(2)} = \int_a^b |a_i(s)||b_k| ds,$$

Where the constant $f_i^{(1)}, f_i^{(2)}, a_{ik}^{(1)}$ and $a_{ik}^{(2)}$ ($1 \le i, k \le n$) are known then equations (14) and (15) become respectively

$$c_i - \lambda \sum_{k=1}^{n} a_{ik}^{(1)} c_k = f_i^{(1)}, \qquad i = 1, 2, \dots, n, \qquad \dots (16)$$

$$d_i - \lambda \sum_{k=1}^{n} a_{ik}^{(2)} d_k = f_i^{(2)}, \qquad i = 1, 2, \dots, n, \qquad \dots (17)$$

which are two systems of n algebraic Equations. For the unknowns c_i and d_i Therefore the problem reduces to finding the quantities c_i and d_i, $i = 1, 2, \dots, n$. The determinants of these systems are two polynomials in term of λ of degree at most n. For all values of λ for which the determinants are not equal to zero the algebraic systems (16) and (17) have solution and thereby the fuzzy integral Equations (2) have unique solutions. We can use equations (12) and (13) and obtain the fuzzy solution of the problem.

Numerical illustration:

This section presents numerical illustration on linear Fredholm fuzzy integral equations system and obtained results compared with the exact solutions obtained in non fuzzy (crisp) environment.

Example: Consider the system of Fredholm fuzzy integral equations with:

$$\overline{f_1}(t,r) = \frac{14t^2(r-2)}{3} + \frac{3t^2(r^3-2)}{4} - t(r-2) + \frac{9rt^2(r^4+2)}{4},$$

$$\underline{f_1}(t,r) = rt - \frac{27t^2(r^3-2)}{4} - \frac{14rt^2}{3} - \frac{rt^2(r^4+2)}{4},$$

$$\overline{f_2}(t,r) = \frac{8(t^2+1)(r-2)}{3} - t(3r^3-6) + \frac{9(r^3-2)(t-2)^2}{10} + \frac{47r(r^4+2)(t-2)^2}{10},$$

$$\underline{f_2}(t,r)=t\left(r^5+2r\right)-\frac{141\left(r^3-2\right)(t-2)^2}{10}-\frac{8r\left(t^2+1\right)}{3}-\frac{3r\left(r^4+2\right)(t-2)^2}{10},$$

kernel functions

$$K_{1,1}(s,t)=t^2(1+s),\quad K_{1,2}(s,t)=t^2\left(1-s^2\right),$$

$$K_{2,1}(s,t)=s\left(1+t^2\right),\quad K_{2,2}(s,t)=(t-2)^2\left(1-s^3\right),\ 0\le s,t\le 2\ ,$$

and a = 0, b = 2, $\lambda_{i,j}$ = 1 (*for i, j* = 1, 2).

The solution of above kernel by crisp method is:

For $K_{1,1}(s,t)=t^2(1+s)$ from equation (2),

$$F(t)=f(t)+\lambda\int_0^2 t^2(1+s)F(s)ds$$

$$F(t)=f(t)+\lambda\int_0^2 \frac{3t^2(1+s)f(s)}{3-20\lambda}ds\,.$$

For $K_{1,2}(s,t)=t^2\left(1-s^2\right)$

$$F(t)=f(t)+\lambda\int_0^2 \frac{5t^2\left(1-s^2\right)f(s)}{(5+8\lambda)}ds\,.$$

For $K_{2,1}(s,t)=s\left(1+t^2\right)$

$$F(t)=f(t)+\lambda\int_0^2 \frac{1+t^2}{1-6\lambda}sf(s)ds\,.$$

For $K_{2,2}(s,t)=(t-2)^2\left(1-s^3\right)$

$$F(t)=f(t)+\lambda\int_0^2 \frac{5\left(1-s^2\right)(t-2)^2}{5-8\lambda}f(s)ds\,.$$

The exact solution in this case is given by

$$\overline{F_1}(t,r)=t(2-r),\qquad \underline{F_1}(t,r)=tr\,,$$

$$\overline{F_2}(t,r)=t\left(6-3r^3\right),\quad \underline{F_2}(t,r)=t\left(r^5+2r\right)$$

In this example we assume that $z = 0$. Using Equations (10)-(11), the coefficients matrix W is calculated as following:

$$W=\begin{bmatrix} W^{1,1} & W^{1,2} \\ W^{2,1} & W^{2,2} \end{bmatrix},$$

where

$$W^{1,1}=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix};\quad W^{1,2}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W^{2,1}=\begin{bmatrix} 2 & \frac{8}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \frac{8}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \quad W^{2,2}=\begin{bmatrix} 2 & \frac{12}{10} & -11 & \frac{-94}{5} \\ -3 & \frac{-22}{10} & 11 & \frac{94}{5} \\ -11 & \frac{-94}{5} & 2 & \frac{12}{10} \\ 11 & \frac{94}{5} & -3 & \frac{-22}{10} \end{bmatrix}$$

With using of above matrices, we can rewrite the linear system (11) as follows:

$$W\begin{bmatrix} \underline{F_1}(0,r) \\ \underline{F_1'}(0,r) \\ \overline{F_1}(0,r) \\ \overline{F_1'}(0,r) \\ \underline{F_2}(0,r) \\ \underline{F_2'}(0,r) \\ \overline{F_2}(0,r) \\ \overline{F_2'}(0,r) \end{bmatrix}=\begin{bmatrix} 0 \\ -r \\ 0 \\ r-2 \\ \frac{6}{5}r^5+\frac{282}{5}r^3+\frac{76}{15}r-\frac{564}{5} \\ -\frac{11}{5}r^5-\frac{282}{5}r^3-\frac{22}{5}r+\frac{564}{5} \\ -\frac{94}{5}r^5-\frac{18}{5}r^3-\frac{604}{15}r+\frac{188}{15} \\ \frac{94}{5}r^5+\frac{3}{5}r^3+\frac{188}{5}r-\frac{66}{5} \end{bmatrix}$$

The analytical vector solution of above linear system is:

$$\begin{bmatrix} \underline{F}_1(0,r) \\ \underline{F}_1'(0,r) \\ \overline{F}_1(0,r) \\ \overline{F}_1'(0,r) \\ \underline{F}_2(0,r) \\ \underline{F}_2'(0,r) \\ \overline{F}_2(0,r) \\ \overline{F}_2'(0,r) \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ 0 \\ 2-r \\ 0 \\ r^5+2r \\ 0 \\ 6-3r^3 \end{bmatrix}$$

CONCLUSION

Proposed method had shown significant role in solving Fredholm equation in terms accuracy and exactness.

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