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SOME RESULTS ON CIRCULAR PERFECT GRAPHS

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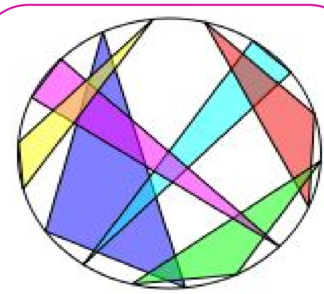
ABSTRACT

An r-circular coloring of a graph G is a map $f: V \to I$ *,where* I *is the set of open unit intervals of an Euclidean circle of length* r *, such that* $f(u) \cap f(v) = \emptyset$ *whenever* $uv \in E(G)$*. Circular perfect graphs are defined analogously to perfect graphs by means of two parameters, the circular chromatic number and the circular clique number. In this article, we study the properties of circular perfect graphs, necessary and sufficient condition for a graph to be circular perfect.*



KEYWORDS: *circular chromatic , necessary and sufficient condition.*

1 INTRODUCTION

If $G = (V, E)$ is a finite simple graph, where V and E denote the set of vertices and the set of edges of G, respectively. Two vertices u and v are adjacent, denoted by $uv \in E(G)$, if there is an edge in $E(G)$ joining them. A proper subgraph of G is a subgraph which is not the graph G itself. A subgraph H of G is called an induced subgraph if $E(H) = \{uv \mid u, v \in V(H) \text{and} uv \in E(G)\}$. A set S of vertices is deleted from a graph G if S together with all the edges with atleast one end in S is removed from G, and the resulting graph is denoted by $G \setminus S$. The independent number of G is denoted by $\alpha(G)$, is the maximum number of vertices contained in an independent set of G. For a given graphs G and H, a homomorphism from G to H is a mapping f from $V(G)$ to $V(H)$ which preserves the adjacent relations.

An n-coloring of G is a mapping from $V(G)$ to $\{1, 2, 3, , , , , , , n\}$ such that adjacent vertices have distinct colors. The least integer n such that G admits an n-coloring is called the chromatic number of G, and is denoted by $\chi(G)$.

It is easy to see that an n-coloring of G is equivalent to a homomorphism from G to K_n. So, if G contains a subgraph isomorphic to K_l, then $\chi(G) \geq l$. We use $\omega(G)$ to denote the clique number of G, which is the maximum number of vertices contained in a complete subgraph of G. In other words, $\omega(G)$ is the maximum number l such that the complete graph K_l admits a homomorphism to G. It is obvious that $\chi(G) \geq \omega(G)$. But the gap between the two parameters χ and ω can be arbitrarily large.

A graph G is called a perfect graph if $\chi(G) = \omega(G)$ for each induced subgraph H of G, and a graph is called imperfect if it is not perfect. The perfect graphs have been extensively studied [2]. The famous perfect Graph conjecture(PGC) [3] claims that a graph is perfect if and only if neither itself nor its complement contains an induced subgraph isomorphic to an odd cycle of length at least five.

In [4], Zhu introduced the concept of circular clique number of graphs and the concept of circular perfect graphs. In this paper, we study the relation between perfect graphs and circular perfect graphs; some basic properties of circular perfect graphs are proved in section 2. In section 3, we give some necessary and sufficient condition for a graph to be circular perfect.

2 CIRCULAR COLORING AND CIRCULAR PERFECT GRAPHS

An r-circular coloring of a graph G is a mapping ψ which assigns to each vertex u of G an open unit length arc $\psi(u)$ of a circle C of length r, such that whenever $uv \in E(G)$, then $\psi(u) \cap \psi(v) = \varnothing$. A graph G is called r-circular colorable if it admits an r-circular coloring. The circular chromatic number of G, denoted by $\psi_c(G)$, is the least r such that G is r-circular colorable. The concept of circular coloring was first introduced in 1988 by Vince who first named it as star coloring, and it got the current name from Zhu. It was proved elsewhere [8] that $\psi_c(G)$ is always attained at rational number and

$\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, for any finite simple graph G. (1)

This means that $\chi(G) = \lceil \chi_c(G) \rceil$. So circular coloring is a refinement of coloring.

A graph named G_k^d plays important roles in the study of circular coloring which is analogous to what K_l does in the coloring. Given two positive integer's k and d such that $k \geq 2d$, the graph G_k^d is defined as follows:

$$V(G_k^d) = \{v_0, v_1, v_2, \ldots\ldots, v_{k-1}\},$$

$$E(G_k^d) = \{v_i v_j \mid d \leq |j - i| \leq k - d \text{ mod } k\}.$$

It is proved elsewhere that $\chi_c(G_k^d) = \frac{k}{d}$[5,6]. So, if a graph G contains a subgraph which is isomorphic to G_k^d for some k and d, then $\chi_c(G) \geq \frac{k}{d}$.

In [4], Zhu introduced the concept of circular clique number. The circular clique number of G, denoted by $\omega_c(G)$, is defined as the maximum fractional $\frac{k}{d}$ such that G_k^d admits a homomorphism to G. Zhu proved in [4] that

Lemma 2.1[4] *For any graph G,*

$$\omega(G) \leq \omega_c(G) \leq \omega(G) + 1, \qquad (2)$$

and $\omega_c(G) = \frac{k}{d}$ for some k and d with $\gcd(k, d) = 1$ indicates that G contains an induced subgraph isomorphic to G_k^d, where gcd(k,d) denotes the greatest common divisors of k and d.

Analogously to the concept of perfect graph, Zhu also introduced in [4] the concept of circular perfect graph, and gave some sufficient conditions and necessary conditions for a graph to be circular perfect. A graph G is called circular perfect if $\omega_c(H) = \chi_c(H)$ for each induced subgraph H of G. A subgraph is called circular imperfect if it is not circular perfect.

Lemma 2.2[4] *For any integers $k \geq 2d, G_k^d$ is circular perfect.*

From (1) and (2), one can easily see that each perfect graph has to be circular perfect. One may suggest that the converse is valid also. But unfortunately, there are infinitely many circular perfect graphs are not perfect. The simplest examples are the odd cycles of length at least five. Let k and d be two positive integers with $k \geq 2d$ and $\gcd(k,d)=1$. By lemma2, graph G_k^d is circular perfect. We now show that G_k^d is not perfect. From (2) and the definition of G_k^d, we have $\omega(G_k^d)=\lfloor \frac{k}{d} \rfloor$, and $\alpha(G_k^d)=d$; it is clearly that $k > d\lfloor \frac{k}{d} \rfloor = \alpha(G_k^d)\omega(G_k^d)$. So G_k^d is not perfect because a graph G is perfect if and only if $|V(H)| = \omega(H)\alpha(H)$ for each induced subgraph H of G[7]

The Weak Perfect Graph Theorem[7] asserts the complement of a perfect graph is still perfect. A similar conclusion is not true for circular perfect graph. The following is an example: Let k, l, r and d be positive integers such that $\gcd(k,d)=1$, $l < r < d$ and $k = ld + r, G$ the complement of G_k^d. We will show that G is circular imperfect. First, one can see that $\omega(G)=d$;by(2), this implies that $\omega_c(G) < d+1$. Second $\alpha(G)=l$, so $\chi(G) \geq \lceil \frac{k}{l} \rceil = \lceil \frac{ld+r}{l} \rceil \geq d+2$; by(1), this implies that $\chi_c(G) > d+1$. Therefore $\chi_c(G) \neq \omega_c(G)$, and G cannot be circular perfect.

There is characterization of perfect graph which says

Lemma 2.3[7] *A graph G is perfect if and only if each induced subgraph H of G has an independent set A such that $\omega(H \setminus A) < \omega(H)$.*

The same conclusion is certainly false when we consider circular perfect graphs instead of perfect graphs. Is there an analogous result for circular perfect graph if we replace ω by ω_c? To answer this question, we have the following

Theorem 2.1 *If G is a circular perfect graph, then for each induced subgrahp H of G, there is an independent set A of $V(H)$, such that $\omega_c(H \setminus A) < \omega_c(H)$. The converse may not be true.*

Proof: since G is circular perfect, we have that $\chi_c(H) = \omega_c(H)$ for each induced subgraph H of G.

Let H bean induced subgraph of $\chi_c(H) = r$, c an r-circular coloring of H on Euclidean circle C of length r, and p an arbitrary point on C. Let A be the set of vertices such that each of them receives an open until interval containing p in c.

If $\omega_c(H \setminus A) < \omega_c(H)$, then we have done. Otherwise, we have $\omega_c(H \setminus A) = \omega_c(H)$. If $\chi_c(H \setminus A) < \chi_c(H)$, then $\omega_c(H \setminus A) \leq \omega_c(H) < \chi_c(H) = \omega_c(H)$, a contradiction.

So, we assume that

$$\chi_c(H \setminus A) = \chi_c(H) = r \qquad (3)$$

We will show that $H \setminus A$ admits an r-coloring. For each vertex v of $H \setminus A$, we use $c(v)$ to denote the interval which is assigned to v by c. By cutting an open circle C at point p, we get an open interval which is identified with the interval $I = [0, r)$. Because no vertex of $H \setminus A$ receives an interval containing p in circular coloring c, we can assume that $c(v)$ is identified with an open unit interval $I(v) \subset I$. Let $I(v)$ be the left end of $I(v)$, and let $f(v)=\lfloor l(v)\rfloor$. It is not difficult to verify that f is an $\lfloor r\rfloor$- coloring of $H\setminus A$. By (1), we have $r=\chi_c(H\setminus A)\le\chi(H\setminus A)\le\lfloor r\rfloor\le r$. Therefore, $\chi(H\setminus A)=r$ which implies that r is an integer. Then by (1),(2),(3) and the fact that H is circular perfect, we have

$$\omega(H)=\omega_c(H)=\chi_c(H)=\chi(H).$$

Let f be an r-coloring of H, and A' a set of vertices which receive the same color. Then,

$$\omega_c(H\setminus A')=\chi_c(H\setminus A')\le\chi(H\setminus A')=\chi(H)-1=\omega(H)-1<\omega_c(H).$$

Now, let us see the Peterson graph P. It is not difficult to verify that each of its induced subgraph H has an independent set A such that

$$\omega_c(H\setminus A)<\omega_c(H).$$

But P is not circular perfect because $\chi_c(P)=\chi(P)=3$ and $\omega_c(P)=\frac{5}{2}$.

Theorem 2.2 *If G is circular perfect then forevery vertex x of G, $N_G[x]$ induces a perfect graph.*($N_G[x]$ or $N[x]$ denote the close neighbourhood of x)

3 SOME SUFFICIENT CONDITIONS FOR A GRAPH TO BE CIRCULAR PERFECT

The necessary condition in Theorem 2.2 for a graph to be circular perfect is not sufficient. By adding some further requirements, we can obtain a sufficient condition.

Theorem 3.1*Suppose G is a graph such that for every vertex x of G, $N[x]$ is a perfect graph and $G-N[x]$ is a bipartite graph with no induced P_5 (i.e., a path with 5 vertices). Then G is circular perfect.*

It is easy to see that if G satisfies the condition of Theorem 3.1, then any induced subgraph of G satisfies that condition. Therefore to prove theorem 3.1, it suffices to prove the following:

Theorem 3.2*Suppose G is a graph and for every vertex x of G, $N[x]$ is a perfect graph and $V-N[x]$ is a bipartite graph which contains no induced P_5. Then $\chi_c(G)=\omega_c(G)$.*

The proof of theorem 3.2 is quit long. The following examples shows that these conditions are quite tight in some sense. Let G be the graph as depicted in Figure 1 below. It is easy to verify that $\chi_c(G)=8/3$ and $\omega_c(G)=5/2$. So G is not circular perfect. But G "almost satisfies" the condition of theorems 3.2. For each vertex x of G, $N[x]$ is a star, and $G-N[x]$ is either a P_4 or a P_5.

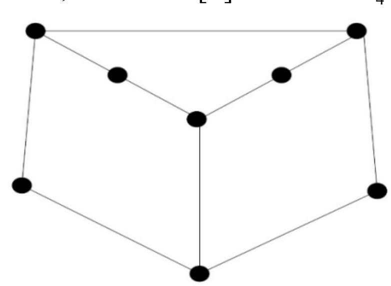


Figure 1: A non-circular perfect graph G

On the other hand, it is easy to construct circular perfect graphs G which contain vertices x such that $G-N[x]$ is not bipartite, or $G-N[x]$ is bipartite but contains induced path length greater than 5. Indeed, if $k=2d+1$ and $d \geq 4$, then the circular complete graph $K_{k/d}$ does not satisfy the condition of theorem 3.1.

The next sufficient condition for a graph to be circular perfect concerns triangle free graphs. Suppose $G=(V,E)$ is a graph and x is a vertex of G. For two vertices $u,v \in V-N[x]$,

- $u \leq^x v$ means $N(u) \cap N(x) \subseteq N(v) \cap N(x)$;
- $u <^x v$ means $N(u) \cap N(x) \subset N(v) \cap N(x)$;
- $u =^x v$ means $N(u) \cap N(x) = N(v) \cap N(x)$.

Here $A \subset B$ means that A is a proper subset of B.

Definition 3.1*Given an induced path $P_n = (p_0, p_1, \ldots\ldots, p_n)$ of $G-N[x]$, we say P_n is badly-linked with respect to x If one of the following holds:*

1. *There are three indices $i<j<k$ of the same parity such that $p_i \not\leq^x p_j$ and $p_k \not\leq^x p_j$.*
2. *There are three indices $i<j<k$ of the same parity such that $p_j \not\leq^x p_i$ and $p_j \not\leq^x p_k$.*
3. *There are three indices $i<j$ and two odd indices $i'<j'$ such that $p_i \not\leq^x p_j$ and $p_{i'} \not\leq^x p_{j'}$.*

An induced path P of $G-N[x]$ is called well-linked with respect to x if it is not badly-linked with respect to x.

Theorem 3.3*Suppose G is a triangle free graph such that for every vertex x of G, $G-N[x]$ is a bipartite graph with no induced C_n for $n \geq 6$, and any induced path of $G-N[x]$ is well-linked. Then G is circular perfect.*

It is easy to see that if G satisfies the condition of theorem3.3, then any induced subgraph of G satisfies that condition. Therefore to prove theorem 3.3, it suffices to prove the following:

Theorem 3.4*Suppose G is a triangle free graph. If for every vertex x of G, $V-N[x]$ is a bipartite graph which contains no induced C_n for $n \geq 6$, and every induced path of $G-N[x]$ ix well-linked, then $\chi_c(G) = \omega_c(G)$.*

The proof of theorem 3.4 is again quite complicated. Theorems 3.1 and 3.3 are used toprove an analogue of *Hajòs* Theorem for circular chromatic number. In [9], three graph operations are given to play the role of the *Hajòs* sum in original *Hajòs* Theorem. Namely it was proved in [9] that if $k/d \geq 3$, then all the graphs G with $\chi_c(G) \geq k/d$ can be constructed from $K_{k/d}$ by adding edges and vertices, by identifying non-adjacent vertices and by applying the three graph operations that replace the *Hajòs* sum. Moreover all such constructed graphs G have $\chi_c(G) \geq k/d$. The same result is proved for $2 \leq k/d < 3$ in [10], where four new graph operations are introduced in place of the *Hajòs* sum.

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