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ON THE CONVEX AND CONCAVE SOLUTIONS OF A STEADY MHD BOUNDARY LAYER EQUATION

I. M. Chandarki[1] and B. B.Singh[2]

[1]Department of General Science And Engineering , N B Navale Sinhgad College Of Engineering , Solapur (M.S.), India.

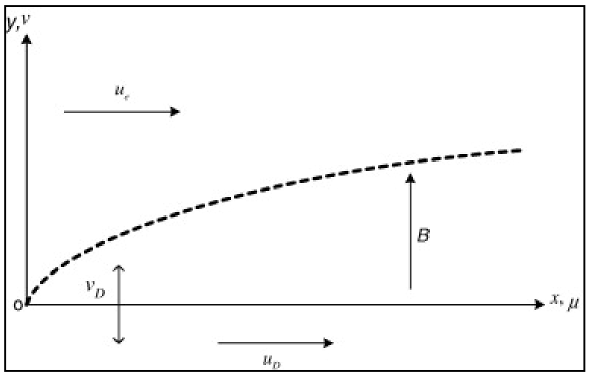
[2]Department of Mathematics, Dr. Babasaheb Ambedkar Technological University, Lonere, Dist.- Raigad (M.S.), India.

ABSTRACT

This paper deals with the convex and concave solutions for a steady laminar incompressible boundary layer equations governing the MHD flow near the forward stagnation point of two-dimensional and axisymmetric bodies. The convex and concave solutions of the boundary layer equations have been discussed in terms of Theorems 1 and 2; the proofs of which are based on topological arguments.

Keywords : *boundary layer equations , topological arguments.*

1. INTRODUCTION



Boundary layer flow of electrically conducting fluid over moving surfaces emerges in a large variety of industrial and technological applications. As a result, it has been investigated by many researchers. Wu [1] has studied the effects of suction or injection on a steady two-dimensional MHD boundary layer flow on a flat plate. Takhar et al. [2] studied a MHD asymmetric flow over a semi-infinite moving surface and numerically obtained the solutions. An analysis of heat and mass transfer characteristics in an electricity conducting fluid over a linearly stretching sheet with variable wall temperature was investigated by Vajruvelu and Rollins [3]. Mahapatra and Gupta [4] treated the steady two-dimensional stagnation point flow of an incompressible viscous electrically conducting fluid towards a stretching surface; the flow being permeated by a uniform transverse magnetic field. For more details, see also [5], [6], [7], [8] and the references therein.

Motivated by the above works, we aim here to give the convex and concave solutions of the third order non-linear autonomous differential equation governing the magneto hydro - dynamic (MHD) flow near the forward stagnation point of two-dimensional and axisymmetric bodies :

$$f'''+\frac{m+1}{2}ff''+m(1-f'^2)+M(1-f')=0 \text{ on } [0, \infty) \qquad ...(1)$$

accompanied by the boundary conditions

$$f(0) = a,\ f'(0) = b,\ f'(\infty) = 1 \qquad ...(2)$$

where a, b, m, M $\in$ R and $f'(\infty) = \lim_{t\to\infty} f'(t)$.

The equation (1) is very interesting because it contains many known equations as particular cases. Setting M = 0 in (1), leads to the well-known Falkner-Skan equation (see [9], [10], [11] and the references therein), while the case M = -m reduces (1) to the equation that arises when considering the mixed convection in a fluid saturated porous medium near a semi-infinite vertical flat plate with prescribed temperature studied by many authors like [12], [13], [14], [15] and the references therein. The case M = m = 0 is referred to the Blasius equation introduced in [16] and studied by several authors (see for example [17], [18], [19]). Recently, the case m = -1 has been studied in [20]. Mention may be made also to the reference [21], where the authors show existence of an infinite number of similarity solutions for the case of a non-Newtonian fluid.

The objective of the present paper is to study the convex and concave solutions of eqns. (1)-(2). As the curvature of the velocity profiles play a significant role in the stability theory (Gortlor [22]), an approach to the study of convex and concave solutions of general similarity boundary layer equation was adopted by Belhachmi et al. [23] and Brighi and Hoernel [24].

2. GOVERNING EQUATIONS

Let us suppose that an electrically conducting fluid, with electrical conductivity σ, in the presence of a transverse magnetic field B(x) is flowing past a flat plate stretched with a power-law velocity. According to [25] and [26], such phenomenon is described by the following equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad ...(3)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_e u_e(x)+\upsilon\frac{\partial^2 u}{\partial y^2}+\frac{\sigma B^2(x)}{\rho}(u_e-u) \qquad ...(4)$$

Here, the induced magnetic field is neglected. In a Cartesian system of co-ordinates (O, x,y), the variables u and v are the velocity components in the x- and y-directions respectively. Here $u_e(x)= \gamma x^m$, $\gamma>0$ denotes the external velocity, B(x), = B_0x^{m-1} the applied magnetic field, m the power-law velocity exponent, ρ the fluid density and υ the kinematic viscosity.

The boundary conditions for the problem (3)-(4) are

$$u(x,0) = u_w(x)=\alpha x^m,\ v(x,0) = v_w(x) = \beta x^{\frac{m-1}{2}},\ u(x, \infty) = u_{ex} \qquad ...(5)$$

Where $u_w(x)$ and $v_w(x)$ are the stretching and the suction (injection) velocity respectively and α, β are constants. Let us recall that $\alpha>0$ is referred to suction, $\alpha<0$ for the injection and $\alpha=0$ for the impermeable plate.

A little inspection shows that the equations (3) and (4) accompanied by conditions (5) admit a similarity solution. Therefore, we introduce the dimensional stream-function ψ in the usual way to get the following equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=u_e u_{ex}+\upsilon\frac{\partial^3\psi}{\partial y^3}+\frac{\sigma B^2(x)}{\rho}(u_e-u) \qquad \text{...(6)}$$

The boundary conditions become

$$\frac{\partial\psi}{\partial y}(x,0)=\alpha x^m,\ \ \frac{\partial\psi}{\partial x}(x,0)=-\beta x^{\frac{m-1}{2}},\ \ \frac{\partial\psi}{\partial y}(x,\infty)=\gamma x^m \qquad \text{...(7)}$$

Defining the similarity variables as follows

$$\psi(x,y)=x^{\frac{m+1}{2}}f(t)\sqrt{\upsilon\gamma} \quad \text{and} \quad t=x^{\frac{m-1}{2}}y\sqrt{\frac{\upsilon}{\gamma}}$$

and substituting in equations (6) and (7) we get the boundary value problem (1)-(2) where

$a=\dfrac{2\beta}{(m+1)\sqrt{\upsilon\gamma}}$, $b=\dfrac{\alpha}{\gamma}$ and $M=\dfrac{\sigma B_0^2}{\gamma\rho}>0$ is the Hartman number and prime is for differentiating with respect to ‘t’.

3. CONVEX AND CONCAVE SOLUTIONS

In order to study the convex and concave solutions of Equations (1)-(2), we consider the following initial value problem

$$\left.\begin{aligned} & f'''+\left(\frac{m+1}{2}\right)ff''+m(1-f'^2)+M(1-f')=0 \\ & f(0)=a,\quad f'(0)=b,\quad f''(0)=c \end{aligned}\right\} \qquad \text{...(8)}$$

with a suitable value of c.

We denote the solution of (8) by f_c and by $[0, T_c)$ its right maximal interval of existence. Next, integrating (1) on $[0,t]$ for $0<t<T_c$, we obtain the identity

$$f_c''(t) - c + \left(\frac{m+1}{2}\right)[f_c(t)f_c'(t) - ab] + (M+m)t - M[f_c(t) - a] = \left(\frac{3m+1}{2}\right)\int_0^t f_c'^2\,dt \qquad ...(9)$$

We also need the following Lemma :

Lemma 1:

If f is a solution of (1) on $[0, T_c)$ such that there exists a point to satisfying $f''(t_0) = 0$ and $f'(t_0)=1$, then $f''(t)=0$ for every $t\in[0, T_c)$.

Proof : Let f be a solution of (1) on $[0, T_c)$ such that $f''(t_0)=0$ and $f'(t_0)=1$ for some $t_0\in[0, T_c)$. Since $g(t) = t-t_0 + f(t_0)$ is a solution of (1) such that $g(t_0) = f(t_0)$, $g'(t_0)=f'(t_0)$ and $g''(t_0)=f''(t_0)$, we obtain g = f and $f''(t) = 0$.

Theorem 1: For $a\in R$ $0\le b<1$, the problem (1)-(2) admits a unique convex solution under the cases (i) $-1<m<0$ and $M \ge -2m$

(ii) $m\ge 0$ and $M > -m(b+1)$

Proof of Existence

Let $f_c(t)$ be a solution of the initial value problem (8) with $0\le b<1$ and $c\ge 0$. We notice that $f_c(t)$ exists as long as we have $f''_c(t)>0$ and $f'_c(t)<1$. From the Lemma, $f''_c(t)$ can not vanish at a point where $f'_c(t)=1$. Therefore, it follows that there are three possibilities :

(a) $f''_c(t)$ becomes negative from a point such that $f'_c(t)<1$,
(b) $f'_c(t)$ takes the value 1 at some point for which $f''_c(t)>1$ and
(c) we always have $0<f'_c(t)<1$ and $f''_c(t)>0$.
(d)

As $f'_0(0) = b<1$, $f''_0(0) = 0$ and $f'''_0(0) = -m(1-b^2)-M(1-b)$

$= -(1-b)\,[m(1+b)+m]<0$ for $m\ge 0$ and $M>-m(b+1)$.

So, we have $f_0(t)$ is of type (a) and by continuity it must be so for $f_c(t)$ with c>0 small enough.

On the otherhand, as long as $f''_c(t)>0$ and $f'_c(t)\le 1$, we have $f_c(t)\le t+a$, and (9) leads to

$$f'_c(t)\ge ct - \frac{1}{2}a(1-b)t + b \qquad ...(10)$$

Hence for c large enough, the polynomial on the right hand side of (10) takes values greater than 1. Therefore, for such a large c, there exists t_0 such that $f'_c(t)=1$ and $f''_c(t)>0$ for $t\le t_0$ and $f_c(t)$ is of type (b).

Defining $A = \{c>0 : f_c(t) \text{ is of type (a) }\}$
and $B = \{c>0 ; f_c(t) \text{ is of type (b) }\}$,

we have that $A\neq\phi$, $B\neq\phi$ and $A\cap B=\phi$. Both A and B are open sets, so there exists a $c^*>0$ such that the solution $f_{c^*}(t)$ of (8) is of type (c), and is defined on the whole interval $[0, \infty)$. For this solution we have that $o<f'_{c^*}(t)<1$ and $f''_{c^*}(t)>0$, which implies that $f'_{c^*}(t)\to\ell\in(0,1]$ as $t\to\infty$.

Let as assume that $\ell\neq 1$; as $f'_{c^*}(t)$ is increasing we have $f'_{c^*}(t)\leq\ell$ and from (1)

$f'''_{c^*}(t)\leq -af''_{c^*}(t)-[2m+M]$ [if $-1<m<0$, $M\geq -2m$ and $\ell<1$]

i.e., $f''_{c^*}(t)\leq -a\{f'_{c^*}(t)-b\}-[2m+M]t+c^*$

As $f'_{c^*}(t)\to\ell<1$ and $f_{c^*}(t)\to\infty$, we obtain a contradiction with the positivity of $f''_{c^*}(t)$.

Proof of Uniqueness

Let f(t) be a convex solution of (1) and (2). As $f(t)>0$ for t large enough, we have that $f'''(t)<0$ for t large enough. Then $f''(t)\to 0$, as $t\to\infty$. As $f'(t)$ and $f''(t)$ are positive, we can define a function $p: [\beta^2,1)\to[\alpha,\infty)$ such that

$\forall\ t\geq 0$; $p(f'(t)^2)=f(t)$.

Setting $y=f'(t)^2$ leads to

$$f(t)=p(y),\quad f''(t)=\frac{1}{2p'(y)}\quad\text{and}\quad f'''(t)=-\frac{p''(y)\sqrt{y}}{2p'(y)^3} \qquad \text{...(11)}$$

and using (1), we obtain

$$p''=(m+1)\frac{p'(y)^2\,p(y)}{\sqrt{y}}+2[m(1-y)+M(1-\sqrt{y})]p'(y)^3 \qquad \text{...(12)}$$

with $p(b^2)=p(f'(0)^2)=a$, $p(1)=\lim_{t\to\infty}f(t)=\infty$ and $p'(b^2)=\frac{1}{2c}>0$.

Let us now suppose that there are two convex solutions $f_1(t)$ and $f_2(t)$ of (1) and (2) with $f''_i(0)=c_i$, $i\in\{1, 2\}$ and $c_1>c_2$. They give p_1, p_2 as solutions of (12) defined on $[b^2, 1)$ such that

$$p_1(b^2)=p_2(b^2)=a,\quad p'_1(b^2)=\frac{1}{2c_1}\ \text{and}\ p'_2(b^2)=\frac{1}{2c_2}$$

Let $\omega=p_1-p_2$; we have $\omega(b^2)=0$ and $\omega'(b^2)<0$. If ω' vanishes, there exists on $x\in[\beta^2,1)$ such that $\omega'(x)=0$, $\omega''(x)\geq 0$ and $\omega(x)<0$.

We then obtain from (12),

$$\omega'' = (m+1)\frac{p_1^2(x)}{\sqrt{x}}\omega(x) < 0 \qquad [\because m+1>0]$$

and this is a contradiction, therefore, $\omega'<0$ and $\omega<0$ on $[b^2,1)$. Let us set now $P_i = \frac{1}{p_i'}$ for $i\in\{1,2\}$ and $W = P_1 - P_2$. We have $W>0$, and using (12), we obtain

$$W'(y) = -(m+1)\frac{\omega(y)}{\sqrt{y}} - 2[m(1-y) + M(1-\sqrt{y})]\omega'(y) > 0$$

But, using (11), we have $P_i(f'(t)^2) = 2f''(t)$ and thus $P_i(y)\to 0$ as $y\to 1$. Hence $W(y)\to 0$ as $y\to 1$, a contradiction. Hence, the convex solution of (1)-(2) is always unique.

Theorem 2: Let $a \in R$ and $b>1$. Then there exists a unique concave solution of the problem (1)-(2) in the following two cases
(i) $-1 < m \le 0$ and $M>-m(b+1)$; (ii) $m > 0$ and $M \ge -2m$.

Proof of existence

Let $f_c(t)$ be a solution of the initial value problem (8) with $b>1$ and $c\le 0$. As long as we have $f_c'(t)<1$ and $f''_c(t)<0$, then $f_c(t)$ exists. Because of lemma 1, there are only three possibilities:
(a) $f''_c(t)$ becomes positive from a point such that $f'_c(t) < 1$;
(b) $f'_c(t)$ takes the value 1 at some point for which $f''_c(t) <0$ and
(c) we always have $1<f''_c(t)$ and $f''_c(t) <0$.
As $f'_0(o) = b>1$, $f''_0(o) = 0$ and $f'''(0) = (b-1)[m(b+1)+M]>0$ if $M > -m(b+1)$, or if $M \ge -2m$, $m>0$, we have that $f'_0(t)>1$ and $f''_0(t)>0$ on some interval $[0, t_0)$. Then, by continuity for small values of $-c$, we have that $f''_c(t)$ becomes positive at some point with $f'_c(t)>1$, and $f_c(t)$ is of type (a).

As long as $f''_c(t) < 0$ and $f'_c(t) \ge 1$, we have $f_c(t) \ge a$ and (9) leads to

$$f''_c(t) \le c + \left\{\left(\frac{3m+1}{2}\right)b^2 - (M+m)\right\}t$$

Hence, for $-c$ large enough, there exists t_0 such that $f'_c(t_0)=1$ and $f''_c(t)<0$ for $t\le t_0$, and $f_c(t)$ is of type (b).

Defining $A = \{c<0 : f_c(t) \text{ is of type (a)}\}$ and $B = \{c<0 : f_c(t) \text{ is of type (b)}\}$, we have that $A \ne \phi$, $B \ne \phi$ and $A \cap B = \phi$. Both A and B are open sets, so there exists a point $c_* < 0$ such that the solution $f_{c*}(t)$ of (8) is of type (c), and is defined in the interval $[0, \infty)$. For this solution, we have that $f'_{c*}(t)>1$ and $f''_{c*}(t)<0$, which implies that $f'_{c*}(t)\to \ell \ge 1$ as $t \to\infty$.

Let us suppose that $\ell \ne 1$; as $f'_{c*}(t)$ is decreasing we have $f'_{c*}(t)\ge \ell$ and from (1)

$$f'''_{c*}(t) \geq -\left(\frac{m+1}{2}\right) \propto f''_{c*}(t) - m(1 - f'^{2}_{c*}) - M(1 - f'_{c*})$$

Integrating this inequality leads to

$$f''_{c*}(t) \geq c_* - \left(\frac{m+1}{2}\right) \propto f'_{c*} - \left[m(1-l^2) + M(1-l)\right] t$$

and, as $f'_{c*}(t) \to \ell$ (≥ 1) as $t \to \infty$, we obtain a contradiction with the negativity of $f''_{c*}(t)$.

Proof of Uniqueness

Let $f_1(t)$ and $f_2(t)$ be two concave solutions of (1) and (2), and let $c_i = f''_i(0) < 0$, $i \in 1,2\}$ with $c_1 > c_2$. Writing $g(t) = f_1(t) - f_2(t)$, we have $g'(0) = 0$, $g'(\infty) = 0$ and $g''(0) > 0$. Hence, $g'(t)$ admits a positive maximum at some point $t_0 > 0$ such that $g'(t) > 0$ on $(0, t_0]$. Therefore, we have

$$g(t_0) > 0,\ g'(t_0) > 0,\ g''(t_0) = 0 \text{ and } g'''(t_0) \leq 0$$

From(1), and since $f'_i(t) > 1$ and $f''_i(t) < 0$, we obtain

$$g'''(t_0) = -\left(\frac{m+1}{2}\right) f''_1(t) g(t_0) - \left(\frac{m+1}{2}\right) g''(t_0) f_2(t_0) + m\{f'_2(t) + f'_1(t)\} g'(t_0) + M g'(t_0) > 0$$ and

hence a contradiction. So, there will always be a unique concave solution under the given conditions.

4. CONCLUDING REMARKS

In this paper the convex and concave solutions for steady laminar incompressible boundary layer equations governing the magneto-hydrodynamic flow near the forward stagnation point of two-dimensional and axisymmetric bodies have been discussed. The curvature of the velocity profiles plays a significant role in the stability of the laminar boundary layer flow. In the immediate neighbourhood of the wall, the curvature of the velocity profiles depends only on pressure gradient, and the curvature of the velocity profile at the wall changes its sign with pressure gradient. For the flow with decreasing pressure (accelerated flow, $\frac{dp}{dx} < 0$), we have that $\frac{\partial^2 u}{\partial y^2} < 0$ (Schlichting [27], p.133) over the whole width of the boundary layer. In the region of pressure increase (decelerated flow, $\frac{dp}{dx} > 0$) we find that $\frac{\partial^2 u}{\partial y^2} > 0$. This shows that the strong dependence of the limit of the stability on the form of velocity profile is equivalent to a great influence of the pressure gradient on the stability. It is seen that the laminar boundary layers in the pressure drop region

$\left(\frac{dp}{dx}<0,\frac{\partial^2 u}{\partial y^2}<0\right)$ are more stable than those in the pressure increase region $\left(\frac{dp}{dx}>0,\frac{\partial u}{\partial y^2}>0\right)$.

So we can conclude from the above discussion that the concave solutions; i.e. by the solutions for which $\frac{\partial^2 u}{\partial y^2}<0$ represent the more stable laminar boundary layers than those represented by convex solutions i.e. by the solutions for which $\frac{\partial^2 u}{\partial y^2}>0$.

From Theorem 1, it is obvious that for (i) –1<m<0, M≥-2m and (ii) m≥0, M>-m(b+1) where a<b<1, the velocity profiles are convex in nature i.e. they will be less stable. They will be more prone to become turbulent. From Theorem 2, it is obvious that for (i) – 1<m≤0, M>-m(b+1) and (ii) m>0, M≥-2m where b>1, the velocity profiles will be concave in nature, i.e. they are more stable and hence they will be less prone to become turbulent.

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