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FIXED POINT THEOREM FOR GENERAL CONTRACTION IN 2-METRIC SPACE.

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ABSTRACT: In this paper we have established sufficient condition for existence of unique fixed point of contraction type mappings on complete 2-metric space for three maps.

KEYWORDS: Fixed Point Theorem , 2-metric space , Euclidean space .

INTRODUCTION

The concept of 2-metric space was initially introduced by Gahler [1] whose abstract properties was suggested by the area of function in Euclidean space. Iseki[2] set out the tradition of proving fixed point theorem in 2-metric spaces employing various contractive conditions. Lal and Singh[3],Rhodes[5]etc. extended the several results of metric space to 2-metric space.
In this paper we have proved sufficient condition for existence of unique fixed point for three maps.

2. PRELIMINARIES:

Now we give some basic definitions and well known results that are needed in the sequel.

Definition (2.1) [1] Let X be a non-empty set and d: X xXxX → R_+. If for allx, y, z, and u in X. We have

(d_1) d(x, y, z) = 0 if at least two of x, y, z are equal.

(d_2) for all x ≠ y, there exists a point z in x such that d(x, y, z) ≠ 0.

(d_3) d(x, y, z) = d(x, z, y) = d(y, z, x) =............... and so on

(d_4) d(x, y, z) ≤ d(x, y, u) + d(x, u, z) + d(u, y, z)

Then d is called a 2-metric on X and the pair (X, d) is called 2-metric space.

Definition (2.2) : A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X,d) is said to be a cauchy sequence

if $\lim_{\substack{n\to\infty \\ m \to \infty}} d(x_n, x_m, a) = 0$ for all a ∈ X.

Definition (2.3) : A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X,d) is said to be a convergent if $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a ∈ X. The point x is called the limit of the sequence.

Definition (2.4) : A 2-metric space (X,d) is said to be complete if every cauchy sequence in X is convergent.

3. MAIN RESULTS

Theorem (3.1) : Let E, F and T be three self maps of a complete 2-metric space (X, d) s.t.

(i) T is continuous,

(ii) {E, T} and {F, T} are commutating pairs,

(iii) E(X) ⊂ T(X) : F(X) ⊂ T(X),

(iv) $d(E^m x, F^n y, a) \le h \max \{d(Tx, Ty, a), d(E^m x, Tx, a), d(F^n y, Ty, a),$
$\frac{1}{2}[d(Tx, F^n y, a)\, d(E^m x, Ty, a)]\}.$

for all x, y, a in X. where $h \in (0,1)$.

Then E, F and T have a unique common fixed point in X.

Proof. Using (ii) and (iii)

$$E^m T = TE^m ;\ F^n T = TF^n$$

and

$$E^m(x) \subset E(x) \subset T(x)$$
$$F^n(x) \subset F(x) \subset T(x)$$

Let x_0 be any arbitrary point of X. Since $E^m(X) \subset T(X)$,

We can choose a point x_1, is X such that $Tx_1 = E^m x_0$. Also $F^n(X) \subset T(X)$, we can choose a point x_2 in such $Tx_2 = F^n x_1$.

In general

$Tx_{2p+1} = E^m x_{2p}$ and $Tx_{2p+2} = F^n x_{2p+1}$, for p = 0, 1, 2, ...

Now first we prove that $d(Tx_{2p},Tx_{2p+1},Tx_{2p+2}) = 0$.

$$d(Tx_{2p},Tx_{2p+1},Tx_{2p+2}) = d(E^m x_{2p}, F^n x_{2p+1}, Tx_{2p})$$
$$\le h\max \{d(Tx_{2p},Tx_{2p+1}, Tx_{2p}), d(E^m x_{2p}, Tx_{2p}, Tx_{2p}),$$
$$d(F^n x_{2p+1},Tx_{2p+1},Tx_{2p}),$$
$$\tfrac{1}{2}[d(x_{2p}, F^n x_{2p},Tx_{2p})+d(E^m x_{2p},Tx_{2p+1}, Tx_{2p}\}$$
$$= h\{d(Tx_{2p+2}, Tx_{2p+1}, Tx_{2p})\}.$$

i.e. $d(Tx_{2p+2}, Tx_{2p+1}, Tx_{2p}) \le hd\, (Tx_{2p+2}, Tx_{2p+1}, Tx_{2p})$ which is not possible.

Hence, $d(Tx_{2p}, Tx_{2p+1}, Tx_{2p+2}) = 0$

Now we consider

$$d(Tx_{2p}, Tx_{2p+1},a) = d(E^m x_{2p-1}, F^n x_{2p}, a)$$
$$\le h \max\{d(Tx_{2p-1}, Tx_{2p}, a), d(E^m x_{2p-1},Tx_{2p-1},a),d(F^n x_{2p}, Tx_{2p},a)$$
$$\tfrac{1}{2}[d(Tx_{2p-1}, F^m x_{2p}, a) + d(E^m x_{2p-1}, Tx_{2p},a)]\}$$
$$= h \max\{d(Tx_{2p-1}, Tx_{2p}, a)\, d(Tx_{2p},Tx_{2p-1}, a), d(Tx_{2p+1}, Tx_{2p}, a),$$
$$\tfrac{1}{2}[d(Tx_{2p-1}, Tx_{2p+1},a) + d(Tx_{2p}, Tx_{2p}, a]\}$$
$$= h \max\{d(Tx_{2p-1}, Tx_{2p}, a), d(Tx_{2p+1}, Tx_{2p},a)\}$$

If $d(Tx_{2p+1}, Tx_{2p}, a) > d(Tx_{2p-1}, Tx_{2p}, a)$, then

$$d(Tx_{2p}, Tx_{2p+1}, a) \le hd(Tx_{2p+1}, Tx_{2p}, a), \text{ a contradiction}$$

Hence, $d(Tx_{2p+1}, Tx_{2p}, a) \le hd(Tx_{2p-1}, Tx_{2p}, a)$

Again

$$d(Tx_{2p+1}, Tx_{2p+2}, a) =d(E^m x_{2p}, F^n x_{2p+1},a).$$
$$\le h \max\{d(Tx_{2p}, Tx_{2p+1}, a), d(E^m x_{2p}, Tx_{2p}, a), d(F^n x_{2p+1}, Tx_{2p+1},a)$$
$$\tfrac{1}{2} [d(Tx_{2p}, F^n x_{2p+1}, a) + d(E^m x_{2p}, Tx_{2p+1},a)]\}.$$
$$=h \max\{d(Tx_{2p}, Tx_{2p+1}, a), d(Tx_{2p+1}, Tx_{2p}, a), d(Tx_{2p+2}, Tx_{2p+1},a)\}$$
$$1/2[d(Tx_{2p}, Tx_{2p+2},a) +d(Tx_{2p+1}, Tx_{2p+1},a)]\}.$$

$=h \max.\{d(Tx_{2p}, Tx_{2p+1},a), d(Tx_{2p+2}, Tx_{2p+1},a)\}.$

If $d(Tx_{2p+2}, Tx_{2p+1},a) > d(Tx_{2p+1}, Tx_{2p}, a)$, then

$d(Tx_{2p+2}, Tx_{2p+1},a) \le hd(Tx_{2p+2}, Tx_{2p+1}, a)$, a contradiction

Hence, $d(Tx_{2p+1}, Tx_{2p}, a) \le hd(Tx_{2p+1}, Tx_{2p}, a)$

$$\le h^2 d(Tx_{2p}, Tx_{2p-1}, a)$$
$$\vdots$$
$$\le h^{2n} d(Tx_o, Tx_1, a)$$

hence, $\{Tx_{2p}\}$ is convergent. Let x be the limit point of this sequence.

Now,

$$E^m Tx_{2p} = TE^m x_{2p} = Tx$$
$$F^n Tx_{2p+1} = TF^n x_{2p+1} = Tx.$$

Now we show that $E^m x = Tx = F^n x$.

$d(E^m x, Tx, a) \le d(E^m x, Tx, TTx_{2p+2}) + d(E^m x, T. Tx_{2p+2},a) + d(TTx_{2p+2},Tx,a)$

$\le d(E^m x, Tx, TTx_{2p+2}) + d(TTx_{2p+2}, Tx,a)$

$+ h \max.\{d(Tx, TTx_{2p+2},a), d(E^m x, TTx_{2p+2},a), d(F^n Tx_{2p+2}, TTx_{2p+2},a),$

$½[d(Tx, F^n Tx_{2p+2},a) + d(E^m x, TTx_{2p+2}, a)]\}.$

when $p \to 0$ $TTx_{2p+1} \to Tx$, $E^m Tx_{2p} = TE^m x_{2p} \to Tx$, $F^n Tx_{2p+1} = TF^n x_{2p+1} \to Tx$

$d(E^m x, Tx,a) \le h \max\{d(E^m x, Tx, a), ½ [d(E^m x, Tx, a)]\}.$

or, $d(E^m x, Tx, a) \le hd (E^m x, Tx, a)$, which is a contradiction.

Thus, $d(E^m x, Tx, a) = 0$ which gives $E^m x = Tx$. Similarly $F^n x = Tx$.

Also $E^m x = x = Tx = F^n x$ as follows.

$d(E^m x, x, a) \le d(E^m x, x, Tx_{2p+2}) + d(E^m x, Tx_{2p+2}, a) + d(Tx_{2p+2}, x,a)$

$\le d(E^m x,x,Tx_{2p+2})+d(Tx_{2p+2},x,a)+h \max\{d(Tx,Tx_{2p+1},a),$

$d(E^m x,Tx,a),d(F^n x_{2p+1},Tx_{2p+1}, a), ½ [d(E^m x,Tx_{2p+1},a),d(F^n x_{2p+1},Tx_, ,a)]$

when $p\to 0$,$Tx_{2p+2}\to x$, $Tx_{2p+1}\to x$ and $E^m x = Tx$, we have

$d(E^m, x, x, a) \le hd (E^m x,x,a)$, which is a contradiction

So, $d(E^m x, x, a) = 0$ which gives $E^m x = x$, similarly $F^n x = x$.

Also $E^m x = Tx = x$ since $E^m x = Tx$. Therefore $Tx = x$.

Thus we get $E^m x = F^n x = Tx = x$.

Now

$TEx = ETx = Ex = E(E^m x) = E^m (Ex).$

i.e. Ex is a common fixed point of T and E^m. Similarly Fx is a common fixed point of T and F^n. But x is a unique common fixed point of E^m, F^n and T.

Hence, Ex = Tx = x = Fx.

Now we shall prove that x is a unique fixed point of E, F and T.

If possible let y is another common fixed point of E, F and T.

Then, Ey = Fy = Ty = y.

Then, $d(x, y, a) = d(Ex, Fy, a) = d(E^m x, F^n y, a)$

$$\leq h \max\{d(Tx, Ty, a), d(E^m x, Tx, a), d(F^n y, Ty, a), \tfrac{1}{2}\{d(Tx, F^n y, a) + d(E^m x, Ty, a)]\}.$$

or, $d(x, y, a) \leq 0$, which is a contradiction and hence x = y.
Thus x is the unique common fixed point of E, F and T.//

Remarks:
Putting T = I, identity mapping in above theorem we get

$$d(E^m x, F^n y, a) = h \max\{d(x, y, a), d(E^m x,x,a), d(F^n y, y, a), \tfrac{1}{2}[d(x, F^n y,a) + d(E^m x, y, a)]\}.$$

which is the result of Rhoades [5]. Hence we generalizes the result of Rhoades.[5].

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