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# UNSTEADY AND STEADY STOKES EQUATIONS: FUNDAMENTAL SOLUTIONS OF BRINKMAN EQUATIONS AND OSEEN EQUATIONS

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**Abstract:**

The Navier-Stokes equations assume an important place in describing the flow of fluids. These equations are derived from the principle of conservation of linear momentum and conservation of mass. The Navier-Stokes equations 111 for an incompressible fluid are given by

$$\rho\frac{\partial V}{\partial t} + \rho V \cdot \nabla V = -\nabla p + \mu \nabla^2 V + \rho f,$$

$$\nabla \cdot \mathbf{V} = 0,$$

where v. is the fluid velocity, p is the density, μ is the coefficient of dynamic viscosity of the fluid and f is the body force per unit mass. The Navier-Stokes equations are useful for a wide range of practical problems but have some limitations. It is difficult to find exact solutions in many flow problems due to the nonlinear nature of these equations. Instead of Navier Stokes equations, some approximations of these equations are sometimes used,

$$\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)(\mathrm{Curl}\,V) = 0.$$

depending on the nature of the flows. Each flow has a characteristic velocity U and a spatial variation on a length scale L.

**keywords :**
Fundamental Solutions , Unsteady and Steady Stokes Equations.

## INTRODUCTION :

The ratio of the inertial and viscous forces defines a dimensionless quantity and is termed as the Reynolds number and is denoted by Re where

$$\mathrm{Re} = \frac{\rho L U}{\mu}.$$

When the viscous forces are more dominant compared to the inertial forces, the Reynolds number is low. In such flows, we neglect the nonlinear inertial convective terms in the Navier Stokes equations, which result in simpler linear equations called Stokes equations.

The corresponding flows are called Stokes flows or creeping flows or flows at low Reynolds number. The physical meaning of low Reynolds number flow or creeping flow is that at least one of the following holds: (i) small length scales (ii) high viscous fluids (iii) very small velocities. The hydromechanics of low Reynolds number flows play a dominant role in the study of theology, lubrication theory, micro-organism locomotion and many bio-physical and geophysical subjects. The study of Stokes[2] equations has always attracted researchers due to their innumerable applications in science and industry. Unsteady and steady Stokes flows have been an important subject of study for mathematicians as well as engineers leading to an interesting mathematical theory and also a wide variety of applications.

Another area of fluid dynamics that has drawn considerable interest of scientists and engineers is the flow through porous media due to numerous applications in science and industry. The Darcy equationst31 and Brinkman equationst41 are two commonly used models employed for low and high porosity systems. The Brinkman equations can be considered to be a modification of Darcy equations and account for the balance of viscous forces and damping forces. In fact, Brinkman equations approximate to the Darcy equations for small values of permeability and to the. Stokes equations for large values of permeability. Another approximation which is also obtained by a process of linearization of the Navier-Stokes equations, resulting in what are known as Oseen equationsf51, is also sometimes used. We briefly describe some aspects of the unsteady and steady Stokes equations, Brinkman equations and Oseen equations in the following few sections.

## 1 Unsteady and Steady Stokes Equations

### Unsteady Stokes equations:

The equations of motion for the unsteady Stokes flow of an incompressible fluid in the absence of any external forces are given by

$$\rho\frac{\partial \mathbf{V}}{\partial t} = -\nabla p + \mu\nabla^2\mathbf{V}, \qquad (1.1)$$

$$\nabla\cdot\mathbf{V} = 0. \qquad (1.2)$$

We rewrite equation (1.1) as

$$\mu\left(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\mathbf{V} = \nabla p, \qquad (1.3)$$

where $\nu = (\mu/\rho)$ is the coefficient of kinematic viscosity. From the above equations (1.1) and (1.2), we get the following.

1. By taking divergence of (1.1) and using equation (1.2), we find that pressure is harmonic.
2. On operating $\Delta^2$on both sides of (1.3), we can verify that any solution V of (1.1) and (1.2) satisfies the equation

$$\nabla^2(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t})\mathbf{V} = 0. \qquad (1.4)$$

3.On operating Curl on either sides of equation (1.1), we find that the vorticity Cur1V satisfies the equation

$$(\nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t})(\mathrm{Curl}\mathbf{V}) = 0. \qquad (1.5)$$

### Steady Stokes equations:

The equations of motion for the steady flow of an incompressible, viscous fluid in the absence of external forces at low Reynolds number are

$$\mu\nabla^2\mathbf{V} = \nabla p, \quad (1.6)$$

$$\nabla\cdot\mathbf{V} = 0, \quad (1.7)$$

We observe from the equations (1.6) and (1.7) that p and Cur1V are harmonic and V is biharmonic.

## 2 Brinkman Equations

Brinkman's equations[4] were originated to estimate the permeability of porous media. The Brinkman equations are given by

$$\mu\nabla^2\mathbf{V} - \frac{\mu}{k}\mathbf{V} = \nabla p, \quad (2.1)$$

$$\nabla\cdot\mathbf{V} = 0, \quad (2.2)$$

where k is the coefficient of permeability of the medium. Let (V, p) be a solution of Brinkman equations. It can be observed that p is harmonic and V satisfies the equation

$$\nabla^2(\nabla^2 - \lambda^2)\mathbf{V} = 0. \quad (2.3)$$

Where $\lambda^2 = \frac{1}{k}$ We also note that CurIV satisfies the equation

$$(\nabla^2 - \lambda^2)\mathrm{Curl}\mathbf{V} = 0. \quad (2.4)$$

## 3 OSEEN EQUATIONS

Stokes[2] equations have been studied elaborately by researchers due to their innumerable applications in science and industry. It has been observed that the problem of a uniform flow past an infinite cylinder cannot be solved exactly. This means no solution to the Stokes equations can be found which satisfies the no slip boundary conditions on the surface of the cylinder and also the condition of uniform flow at infinity. This is known as the Stokes paradox. This Stokes paradox was solved by Oseent51 in 1910. The limitations of Stokes flow in providing a satisfactory explanation of Stokes paradox or in, explaining the validity of Stokes equations at sufficiently large -distances from an obstacle have prompted Oseen [5] to propose what are now commonly known as ()seen equations. It was suggested by him that the inertia terms should be retained -in the far field where the velocity is approximately equal to U$\hat{k}$( where U > 0 and $\hat{k}$ is the unit vector in the fixed z-direction. These equations too have been studied extensively.

The equations governing the steady Oseen [5] flow of an incompressible fluid are given by

$$\rho U \frac{\partial \mathbf{V}}{\partial z} = -\nabla p + \mu \nabla^2 \mathbf{V}, \qquad (3.1)$$

$$\nabla \cdot \mathbf{V} = 0, \qquad (3.2)$$

These equations are obtained by linearizing the Navier-Stokes equations by taking V' = V + $U\hat{k}$ and neglecting terms of the second order in the velocity components. We rewrite equation (3.1) as

$$\mu(\nabla^2 \mathbf{V} - 2k \frac{\partial \mathbf{V}}{\partial z}) = \nabla p, \qquad (3.3)$$

where k = U/2v It can be observed that pressure p is harmonic and V satisfies the equation

$$\nabla^2(\nabla^2 - 2k \frac{\partial}{\partial z})\mathbf{V} = 0. \qquad (3.4)$$

We also note that Curl V satisfies the equation

$$(\nabla^2 - 2k \frac{\partial}{\partial z})(\mathrm{Curl}\mathbf{V}) = 0. \qquad (3.5)$$

We can solve the above stated equations of motion using analytical or numerical methods. Few analytical methods are available for solving the above stated equations of motion. Chadwick et al. [6] gave a method of representing any divergence free (solenoidal) vec¬tor field V expressed in terms of two scalar functions. This representation had been exploited by Padmavathi et al. [7], [8] to introduce certain general solutions of Stokes[7] and Brinkman [8] equations. This representation has the following advantages: it is simple to use, the scalars can be found very easily for many types of fluid flows and the boundary conditions formulated in terms of these scalars are in a very simple form. The complexity of the equations is reduced, since these scalar functions satisfy simple partial differential equations, whose solutions are in general known. We shall solve the above stated equations of motion based on the fact that velocity is divergence free (solenoidal) in them.

A solution of the equations of motion which is such that every other solution can be obtained from it is said to be a complete general solution. In this thesis, we discuss the complete general solutions of homogeneous and non-homogeneous unsteady Stokes equations which are suitable for boundary value problems dealing with spherical bound¬aries and also discuss some complete general solutions of Stokes, Brinkman and Oseen equations which are suitable for discussing flow problems dealing with plane boundaries.

## 4 FUNDAMENTAL SOLUTIONS

The fundamental solution of (1.6) and (1.7) which represents a point force was initially obtained by Oseen [5] and was named as 'Stokeslet' by Hancock [14]. The Curl of this fundamental solution is called a 'rotlet'. By taking the directional derivatives of these singular solutions, other higher order singular solutions can obtained (Chwang and Wu[15]).

**Stokeslet:** Consider the non-homogeneous Stokes equations in the presence of an exter¬nal force $f_s$ (r)

$$\mu \nabla^2 \mathbf{V} + \mathbf{f}_s(\mathbf{r}) = \nabla p, \qquad (4.1)$$

$$\nabla \cdot \mathbf{V} = 0, \qquad (4.2)$$

where r is the position vector. Then the primary fundamental solution is associated with a singular point force located at the origin f, = $8\pi\mu\boldsymbol{\alpha}\delta$(r), $\boldsymbol{\alpha}$ being a constant vector and $\delta$(r) the three dimensional Dirac delta-function. We denote the velocity, pressure and vorticity corresponding to this solution by $V_s$, $p_s$ and $\xi_s$ given by

$$\mathbf{V}_S(\mathbf{r};\boldsymbol{\alpha}) = \frac{\boldsymbol{\alpha}}{r} + (\boldsymbol{\alpha}.\mathbf{r})\frac{\mathbf{r}}{r^3}, \qquad (4.3)$$

$$p_S(\mathbf{r};\boldsymbol{\alpha}) = -2\mu\nabla \cdot \frac{\boldsymbol{\alpha}}{r} = 2\mu\boldsymbol{\alpha} \cdot \frac{\mathbf{r}}{r^3}, \qquad (4.4)$$

$$\boldsymbol{\xi}_S(\mathbf{r};\boldsymbol{\alpha}) = 2\mu\nabla \times \frac{\boldsymbol{\alpha}}{r} = 2\boldsymbol{\alpha} \times \frac{\mathbf{r}}{r^3}. \qquad (4.5)$$

Using the expression for Stokeslet we can find [15] expressions for higher order singularities Stokes doublet, Stokes quadruple, rotlet, stress-let and potential doublet.

$$\mathbf{V}_{SD}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta}) = -(\boldsymbol{\beta}\cdot\nabla)\mathbf{V}_S(\mathbf{r},\boldsymbol{\alpha}), \qquad (4:6)$$

$$p_{SD}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta}) = -(\boldsymbol{\beta}\cdot\nabla)\mathbf{V}_S(\mathbf{r},\boldsymbol{\alpha}), \qquad (4.7)$$

$$\mathbf{V}_{S4}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = (\boldsymbol{\gamma}\cdot\nabla)(\boldsymbol{\beta}\cdot\nabla)\mathbf{V}_S(\mathbf{r},\boldsymbol{\alpha}), \qquad (4.8)$$

$$p_{S4}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}) = (\boldsymbol{\gamma}\cdot\nabla)(\boldsymbol{\beta}\cdot\nabla)\mathbf{V}_S(\mathbf{r},\boldsymbol{\alpha}), \qquad (4.9)$$

where α, β and γ are constant vectors constituting the pole moments and the suffixes SD and S4 are ued to denote the flow quantities of the Stokes doublet and Stokes quadruple respectively.

The antisymmetric component of a Stokes doublet (with respect to an interchange of α and β) is called a rotlet (also called a couplet by Batchelor[1]) and its velocity and pressure are given by

$$\mathbf{V}_R(\mathbf{r};\gamma) = \frac{1}{2}[\mathbf{V}_{SD}(\mathbf{r};\boldsymbol{\beta},\boldsymbol{\alpha}) - \mathbf{V}_{SD}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta})] = \frac{1}{2}\nabla \times \mathbf{V}_S(\mathbf{r};\gamma),$$
$$= \gamma \times \frac{\mathbf{r}}{r^3}, \qquad (\gamma = \boldsymbol{\alpha} \times \boldsymbol{\beta}), \tag{4.10}$$

$$p_R(\mathbf{r};\gamma) = \frac{1}{2}[p_{SD}(\mathbf{r};\boldsymbol{\beta},\boldsymbol{\alpha}) - p_{SD}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta})] = 0. \tag{4.11}$$

The symmetric component of Stokes doublet is called a stresslet. Its velocity and pressure are given by

$$\mathbf{V}_{SS}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta}) = \left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{r^3} + 3\frac{(\boldsymbol{\alpha}\cdot\mathbf{r})(\boldsymbol{\beta}\cdot\mathbf{r})}{r^5}\right]\mathbf{r}, \tag{4.12}$$

$$p_{SS}(\mathbf{r};\boldsymbol{\alpha},\boldsymbol{\beta}) = -2\mu\left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{r^3} + 3\frac{(\boldsymbol{\alpha}\cdot\mathbf{r})(\boldsymbol{\beta}\cdot\mathbf{r})}{r^5}\right] \tag{4.13}$$

The velocity and pressure of a potential doublet are given by

$$\mathbf{V}_D(\mathbf{r},\boldsymbol{\delta}) = \nabla(\nabla\cdot\frac{\boldsymbol{\delta}}{r}) = \frac{\boldsymbol{\delta}}{r^3} + 3(\boldsymbol{\delta}\cdot\mathbf{r})\frac{\mathbf{r}}{r^5} = -\frac{1}{2}\nabla^2\mathbf{V}_S(\mathbf{r},\boldsymbol{\delta}), \tag{4.14}$$

$$p_D(\mathbf{r},\boldsymbol{\delta}) = \frac{1}{2}\nabla^2 p_S(\mathbf{r},\boldsymbol{\delta}). \tag{4.15}$$

where δ is the doublet strength.

## UNSTEADY STOKESLET :

The fundamental solution of (1.1) and (1.2) which represents a point force given in[16] is called as an 'unsteady Stokeslet'. Consider the velocity field due to an unsteady point force Fi located at the origin. Then the velocity components for an unsteady Stokeslet in cartesian coordinates are given by

$$u = \frac{F}{8\pi\mu}\left(X(R)\frac{1}{r} + Y(R)\frac{x^2}{r^3}\right)e^{i\omega t}, \tag{4.16}$$

$$v = \frac{F}{8\pi\mu}\left(Y(R)\frac{xy}{r^3}\right)e^{i\omega t}, \tag{4.17}$$

$$w = \frac{F}{8\pi\mu}\left(Y(R)\frac{xz}{r^3}\right)e^{i\omega t}, \tag{4.18}$$

$$p = -\frac{F}{4\pi}\left(\frac{\partial}{\partial x}(\frac{1}{r})\right)e^{i\omega t}, \tag{4.19}$$

where

$$X(R) = 2e^{-R}(1+\frac{1}{R}+\frac{1}{R^2})-\frac{2}{R^2},$$

$$Y(R) = -2e^{-R}(1+\frac{3}{R}+\frac{3}{R^2})+\frac{6}{R^2},$$

$$R = \lambda r, \ \lambda=\frac{\imath\omega}{\nu}.$$

## Oseenlet

The fundamental solution of (3.1) and (3.2) which represents a point force was obtained by Oseen (5] and was named as `Oseenlet'. Consider an Oseenlet due to a point force G$\hat{k}$ located at the origin. Then the velocity components for an Oseenlet in cartesian coordinates are[17]

$$u = \frac{-G}{4\pi\rho U}\frac{\partial}{\partial x}\left(\frac{1}{r}-\frac{e^{-k(r-z)}}{r}\right), \quad (4.20)$$

$$v = \frac{-G}{4\pi\rho U}\frac{\partial}{\partial y}\left(\frac{1}{r}-\frac{e^{-k(r-z)}}{r}\right), \quad (4.21)$$

$$w = \frac{-G}{4\pi\rho U}\left[\frac{\partial}{\partial z}\left(\frac{1}{r}-\frac{e^{-k(r-z)}}{r}\right)+\frac{2k}{r}e^{-k(r-z)}\right], \quad (4.22)$$

where $r^2=(x^2+y^2+z^2)^{1/2}.$

$$p=-\frac{1}{4\pi}\frac{\partial}{\partial z}(\frac{1}{r}). \quad (4.23)$$

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