REVIEW OF RESEARCH
ISSN: 2249-894X
IMPACT FACTOR : 5.7631(UIF)
VOLUME - 12 | ISSUE - 11 | AUGUST - 2023

STOCHASTIC CONTROLLABILITY OF NONLINEAR INTEGRODIFFERENTIAL SYSTEMS WITH TIME VARIABLE DELAY IN CONTROL

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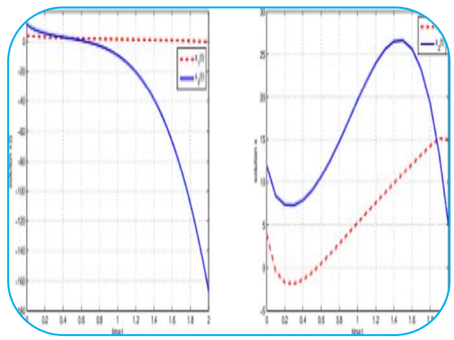
ABSTRACT

We consider the stochastic controllability problem for nonlinear intrgrodifferential systems with time variable delay in control. Controllability problem for nonlinear stochastic differential systems has been studied by enormous researchers. Most of the researchers have considered the nonlinear stochastic differential systems with fixed/variable time delay in control. Some of the researchers have studied controllability of deterministic nonlinear systems with fixed/variable time delay in control.

In this article we have studied the controllability of nonlinear stochastic intrgrodifferential systems with time variable delay in control. We have illustreted the results obtained in this article with some examples.



KEY WORDS: *Controllability; Nonlinear integrodifferential systems; Time variable delay; Stochastic systems.*

I. INTRODUCTION

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{U}$ be separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $\mathbb{E}(\cdot)$ denotes the expectation with respect to measure $\mathbb{P}$. Consider the nonlinear system with variable delay in control as follows:

$$\begin{aligned} dx(t) = \quad & [Ax(t) + B_1 u(t) + B_2 u(v(t)) + f(t, x(t), \int_0^t g(s, u(s))ds)]dt \\ & + h(t, x(t), \int_0^t k(s, u(s))ds)dw(t), \quad t \in I = [0, T], \end{aligned} \tag{1}$$

with initial conditions

$$x(0) = x_0 \in \mathcal{L}_2(\Omega, \mathcal{F}_0, \mathcal{H}) \quad \text{and} \quad u(t) = 0, \quad t \in [v(0), 0]. \tag{2}$$

$A: D(A) \subset \mathcal{H} \to \mathcal{H}$ is a closed linear operator generating strongly continuous semigroup $S(t)$, B_1, $B_2 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ are bounded linear operators and $\{w(t): t > 0\}$ is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q > 0$.

Let $\mathcal{L}_Q(\mathcal{K}, \mathcal{H})$ be the space of all Q-Hilbert-Schmidt operators $h: [0, T] \times \mathcal{H} \times \mathcal{U} \to \mathcal{L}_Q(\mathcal{K}, h)$ with norm $\|\cdot\|$ defined as $\| h \|_Q = Tr(hQh^T)$ and $f: [0, T] \times \mathcal{H} \times \mathcal{U} \to \mathcal{H}$, $g, k: I \times \mathcal{U} \to \mathcal{U}$. $\mathcal{L}_2^{\mathcal{F}}([0, T] \times \mathcal{H}, \mathcal{H})$ is the space of all $\mathcal{F}_t$-adapted, $\mathcal{H}$-valued measurable square integrable processes with the norm $\|\cdot\|_{\mathcal{H}}$. $v(t) = t - \gamma(t)$ is continuously differentiable and strictly increasing function defined on $[0, T]$, and $\gamma(t) > 0$ is a time variable point delay. For convenience, consider the time-leading function $r(t) = t + \gamma(t)$, which is the inverse function for $v(t)$. That is, we have $r(v(t)) = t$. Note that for $t \in [0, v(T)]$ system (1) is in fact a system without delay. Hence, from this point onwards we will assume that $v(T) > 0$.

In this article, we will study the approximate controllability of stochastic semilinear integrodifferential infinite dimensional systems (1) with time variable delay in control. In section 2, some preliminaries are discussed. The approximate controllability of the system (1) is studied in section 3. Finally in section 4, some examples are discussed.

II. PRELIMINARIES

For notational convenience, we assume

$$\begin{aligned} Gu(t) &= \int_0^t g(s,u(s))ds \text{ and } F(x,u)(t) = f(t,x(t),Gu(t)), \\ Ku(t) &= \int_0^t k(s,u(s))ds \text{ and } H(x,u)(t) = h(t,x(t),Ku(t)). \end{aligned}$$

The mild solution of (1) is defined as

$$\begin{aligned} x(t;x_0,u) = &\ S(t)x_0 + \int_0^t S(t-s)[B_1u(s) + F(x,u)(s)]ds \\ &+ \int_0^t S(t-s)B_2u(v(s))ds + \int_0^t S(t-s)H(x,u)(s)dw(s). \end{aligned} \tag{3}$$

Taking into account the zero initial control for $t \in [v(0), 0]$, the mild solution (3) can be written as

$$\begin{aligned} x(t;x_0,u) = &\ S(t)x_0 + \int_0^{v(t)} [S(t-s)B_1 + S(t-r(s))B_2r'(s)]u(s)ds \\ &+ \int_{v(t)}^t S(t-s)B_1u(s)ds + \int_0^t S(t-s)F(x,u)(s)ds \\ &+ \int_0^t S(t-s)H(x,u)(s)dw(s). \end{aligned} \tag{4}$$

Let $\mathcal{U}_{ad} = \mathcal{L}_2^{\mathcal{F}_t}([0,T] \times \Omega; \mathcal{U})$. Define the set of all reachable states as follows:

$$R_T(T;x_0,u) = \{x(T;x_0,u) | u(\cdot) \in \mathcal{U}_{ad},\ x_0 \in \mathcal{L}_2(\Omega, \mathcal{F}_T, \mathcal{H})\}.$$

Definition 1 *The controlled stochastic system (1) is said to be relatively exactly controllable on* $[0,T]$ *if for every initial condition* $x_0 \in \mathcal{L}_2(\Omega, \mathcal{F}_0, H)$*, there is some control* $u \in \mathcal{U}_{ad}$ *such that* $R(T;x_0,u) = \mathcal{L}_2(\Omega, \mathcal{F}_T, \mathcal{H})$.

Definition 2 *The controlled stochastic system (1) is said to be relatively approximately controllable on* $[0,T]$ *if for every initial condition* $x_0 \in \mathcal{L}_2(\Omega, \mathcal{F}_0, H)$*, there is some control* $u \in \mathcal{U}_{ad}$ *such that* $\overline{R(T;x_0,u)} = \mathcal{L}_2(\Omega, \mathcal{F}_T, \mathcal{H})$.

If $T > 0$ is arbitrarily small, then we say that system is small time relatively exactly controllable, and small time relatively approximately controllable.

Define the linear bounded control operator $L_T \in \mathcal{L}(\mathcal{U}_{ad}, \mathcal{L}_2(\Omega, \mathcal{F}_T, \mathcal{H}))$ as follows:

$$L_Tu = \int_0^{v(T)} (S(T-t)B_1 + S(T-r(s)B_2r'(s))u(s)ds + \int_{v(T)}^T S(T-s)B_1u(s)ds.$$

The adjoint $L_T^*: \mathcal{L}_2(\Omega, \mathcal{F}_T, \mathcal{H}) \to \mathcal{U}_{ad}$ of L_T is

$$\begin{aligned} L_T^* &= (B_1^*S^*(T-t) + B_2^*S^*(T-v(t))r'(t))\mathbb{E}\{\cdot|\mathcal{F}_t\}, \quad t \in [0, v(T)], \\ L_T^* &= B_1^*S^*(T-t)\mathbb{E}\{\cdot|\mathcal{F}_t\}, \quad t \in (v(T), T]. \end{aligned}$$

We observe that

$$R(T;x_0,u)=S(T)x_0+\mathrm{Im}L_T+\int_0^T S(T-s)F(x,u)(s)ds$$
$$+\int_0^T S(T-s)H(x,u)(s)dw(s).$$

The deterministic controllability operator is defined as

$$\Psi_s^T=\int_s^{v(T)}(r'(t)S(T-r(t))B_2B_2^*S^*(T-r(t))r'(t)+S(T-t)B_1B_1^*S^*(T-t))dt$$
$$+\int_{v(T)}^T S(T-t)B_1B_1^*S^*(T-t)dt,\qquad s<v(T),$$
$$\Psi_s^T=\int_s^T S(T-t)B_1B_1^*S^*(T-t)dt,\ s\geq v(T),$$

and linear controllability operator $\Pi_0^T\in\mathcal{L}(\mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H}),\mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H}))$ associated with (1) is defined as

$$\Pi_0^T=\int_0^{v(t)}(r'(t)S(T-r(t))B_2B_2^*S^*(T-r(t)r'(t)+S(T-t)B_1B_1^*S^*(T-t))\mathbb{E}\{\cdot|\mathcal{F}_t\}dt$$
$$+\int_{v(t)}^T S(T-t)B_1B_1^*S^*(T-t)\mathbb{E}\{\cdot|\mathcal{F}_t\}dt.$$

We assume

Hypothesis 1

A. The functions $f,h:[0,T]\times\mathcal{H}\times\mathcal{U}\to\mathcal{H},\ g,k:[0,T]\times\mathcal{U}\to\mathcal{L}_Q(\mathcal{K},\mathcal{H})$ satisfy the Lipschitz condition. That is, there exists some positive constant L such that for all $x_1,\ x_2\in\mathcal{H},\ u_1,\ u_1\in\mathcal{U},\ t\in[0,T]$
$\| f(t,x_1,u_1)-f(t,x_2,u_2)\|^2+\| h(t,x_1,u_1)-h(t,x_2,u_2)\|_Q^2$
$\leq L(\| x_1-x_2\|^2+\| u_1-u_2\|^2),$
$\| g(t,u_1)-g(t,u_2)\|^2+\| k(t,u_1)-k(t,u_2)\|^2\leq L\| u_1-u_2\|^2.$

B. The functions f, g are continuous on $[0,T]\times\mathcal{H}\times\mathcal{U}$, and there exists some positive constants $L>0$ such that for all $x\in\mathcal{H},\ u\in\mathcal{U},\ t\in[0,T]$
$\| f(t,x,u)\|^2+\| h(t,x,u)\|_Q^2\leq L(1+\| x\|^2+\| u\|^2)$
$\| g(t,u)\|^2+\| k(t,u)\|_Q^2\leq L(1+\| u\|^2).$

C. f and h are bounded on $[0,T]\times\mathcal{H}\times\mathcal{U}$.

D. g and k are bounded on $[0,T]\times\mathcal{U}$.

Remark 1 *From Hypothesis 1 (A), We have*

$$\mathbb{E}\| Gu_1(t)-Gu_2(t)\|^2+\mathbb{E}\| Ku_1(t)-Ku_2(t)\|$$
$$=\mathbb{E}\left\|\int_0^t(g(s,u_1(s))-g(s,u_2(s)))ds\right\|^2+\mathbb{E}\left\|\int_0^t(k(s,u_1(s))-k(s,u_2(s)))ds\right\|^2$$
$$\leq\ t\mathbb{E}\int_0^t(\| g(s,u_1(s))-g(s,u_2(s)))\|^2+\| k(s,u_1(s))-k(s,u_2(s))\|^2)ds$$
$$\leq\ Ct\mathbb{E}\int_0^t\| u_1(s)-u_2(s)\|^2\,ds$$
$$\leq\ Ct^2\| u_1-u_2\|^2$$

And from Hypothesis 1 (B), We have

$$\begin{aligned}\mathbb{E}(\| Gu(t) \|^2 + \| Ku(t) \|^2) = \; & \mathbb{E}\left\|\int_0^t g(s,u(s))ds\right\|^2 + \mathbb{E}\left\|\int_0^t k(s,u(s))ds\right\|^2 \\ & \leq t\mathbb{E}\int_0^t (\| g(s,u(s)) \|^2 + \| k(s,u(s)) \|^2)ds \\ & \leq Ct\mathbb{E}\int_0^t (1+\| u(s) \|^2)ds \\ & \leq Ct\left(t + \mathbb{E}\int_0^t \| u \|^2 ds\right)\end{aligned}$$

Hence

$$\| Gu \|^2 + \| Ku \|^2 = \sup_{t\in I}(\| Gu(t)) \|^2 + \| Ku(t) \|^2) \leq CT^2(1+\| u \|^2).$$

Under the hypothesis 1, for any $u \in \mathcal{U}_{ad}$ there exists the unique mild solution to an integral equation (4) [7].

III. Controllability of nonlinear systems

Consider the linear stochastic system corresponding to the nonlinear stochastic system (1) defined as follows:

$$\begin{aligned} dx(t) = \; & [Ax(t) + B_1u(t) + B_2u(v(t)) + \tilde{F}(t)]dt + \tilde{H}(t)dw(t), \; t \in I, \\ x(0) = \; & x_0, \end{aligned} \tag{5}$$

where $\tilde{F}(t) = f(t, \int_0^t g(s)ds)$ and $\tilde{H}(t) = h(t, \int_0^t k(s)ds)$.

The mild solution of the linear system (5) is given by

$$\begin{aligned} x(t; x_0, u) = \; & S(t)x_0 + \int_0^{v(t)} [S(t-s)B_1 + S(t-r(s))B_2r'(s)]u(s)ds \\ & + \int_{v(t)}^t S(t-s)B_1u(s)ds + \int_0^t S(t-s)\tilde{F}(s)ds \\ & + \int_0^t S(t-s)\tilde{H}(s)dw(s). \end{aligned} \tag{6}$$

We have the following representation theorem.

Lemma 1 ([9]) *For any $h \in \mathcal{L}_2(\mathcal{F}_T, \mathcal{H})$ there exists a unique $\phi \in \mathcal{L}_2^{\mathcal{F}}([0,T], \mathcal{L}_Q(\mathcal{K}, \mathcal{H}))$ such that*

$$h = \mathbb{E}h + \int_0^T \phi(s)dw(s). \tag{7}$$

Lemma 2 *For arbitrary $h \in \mathcal{L}_2(\mathcal{F}_T, \mathcal{H})$, $\tilde{F}(\cdot) \in \mathcal{L}_2^{\mathcal{F}}([0,T], \mathcal{H})$, $\tilde{H}(\cdot) \in \mathcal{L}_2^{\mathcal{F}}([0,T], \mathcal{L}_Q(\mathcal{K}, \mathcal{H})$, the control*

$$\begin{aligned} u(t) = \; & B_1^*S^*(T-t)(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0) \\ & - B_1^*S^*(T-t)\int_0^{v(T)} (\alpha + \Psi_s^T)^{-1}S(T-s)\tilde{F}(s)ds \\ & - B_1^*S^*(T-t)\int_0^{v(T)} (\alpha + \Psi_s^T)^{-1}[S(T-s)\tilde{H}(s) - \phi(s)]dw(s), \; t \in [0, v(T)], \end{aligned}$$

and

$$\begin{aligned} u(t) = \; & (B_1^*S^*(T-t) + B_2^*S^*(T-r(t)r'(t))(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0) \\ & -(B_1^*S^*(T-t) + B_2^*S^*(T-r(t)r'(t))\int_{v(T)}^T (\alpha + \Psi_s^T)^{-1}S(T-r(s))\tilde{F}(s)ds \\ & -(B_1^*S^*(T-t) + B_2^*S^*(T-r(t)r'(t))\int_{v(T)}^T (\alpha + \Psi_s^T)^{-1}[S(T-r(s))\tilde{H}(s) - \phi(s)]dw(s), \; t \in [v(T), T], \end{aligned} \tag{8}$$

transfers the system (6) from $x_0 \in \mathcal{H}$ to

$$x(T) = h - \alpha(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0) - \alpha \int_0^T (\alpha + \Psi_s^T)^{-1} S(t-s)\tilde{F}(s)ds$$
$$-\alpha \int_0^T (\alpha + \Psi_s^T)^{-1}[S(t-s)\widetilde{H}(s) - \phi(s)]dw(s) \quad (9)$$

at time T. Here ϕ comes from Lemma 1.

Proof. By substituting (8) and into (6) and using Fubini theorem we get (9) (see Lemma 4 in [12]).

Define the operator $\Phi_\alpha: \mathcal{H} \times \mathcal{U}_{ad} \to \mathcal{H} \times \mathcal{U}_{ad}$ as follows:

$$(z^\alpha(t), \beta^\alpha(t)) = \Phi_\alpha(x,u)(t), \quad (10)$$

where

$$z^\alpha(t) = S(t)x_0 + \int_0^{v(t)} (S(t-s)B_1 + S(t-r(r))B_2 r'(s))\beta^\alpha(s)ds + \int_{v(t)}^t S(t-s)B_1\beta^\alpha(s)ds$$
$$+\int_0^t S(t-s)F(x,u)(s))ds + \int_0^t S(t-s)H(x,u)(s)dw(s),$$

where

$$\beta^\alpha = B_1^* S^*(T-t)(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0)$$
$$-B_1^* S^*(T-t)\int_0^{v(T)} (\alpha + \Psi_s^T)^{-1} S(T-s)F(x,u)(s)ds$$
$$-B_1^* S^*(T-t)\int_0^{v(T)} (\alpha + \Psi_s^T)^{-1}[S(T-s)H(x,u)(s) - \phi(s)]dw(s), \ t \in [0, v(T)],$$

and

$$\beta^\alpha = (B_1^* S^*(T-t) + B_2^* S^*(T - r(t)r'(t))(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0)$$
$$-(B_1^* S^*(T-t) + B_2^* S^*(T - r(t)r'(t))\int_{v(T)}^T (\alpha + \Psi_s^T)^{-1} S(T - r(s))F(x,u)(s)ds$$
$$-(B_1^* S^*(T-t) + B_2^* S^*(T - r(t)r'(t)) \times$$
$$\int_{v(T)}^T (\alpha + \Psi_s^T)^{-1}[S(T - r(s))H(x,u)(s) - \phi(s)]dw(s), \ t \in [v(T), T], \quad (11)$$

The existence of a unique fixed point to the operator is proved in the following theorem.

Theorem 1 *Assume that Hypothesis 1 (A) and (B) are true. Then for any* $\alpha > 0$ *the operator* Φ_α *has a unique fixed point.*

Proof. Proof is similar to the proof of Theorem 3.1 in [11].

Now, we prove the relative approximate controllability of the nonlinear stochastic system (1).

Theorem 2 *Assume that Hypothesis 1 is true and linear stochastic system (5) is relatively approximately controllable, then nonlinear stochastic system (1) is relatively approximately controllable.*

Proof. The proof is similar to the proof of Theorem 7 in [12].

Corollary 1 *Assume that Hypothesis 1 is true. If the semigroup S(t) is analytic and the deterministic linear system corresponding to (5) is relatively approximately controllable on* $[0,T]$ *then the nonlinear stochastic system (1) is relatively approximately controllable on* $[0,T]$.

Proof. It is well known that, the semigroup $S(t)$ is analytic and the linear stochastic system (5) is relatively approximately controllable on $[0,T]$ if and only if the deterministic linear system corresponding to (5) is relatively approximately controllable on $[0,T]$ (See [9], Theorem 4.3). Then by Theorem 2 nonlinear stochastic system is relatively approximately controllable.

Define the operator $\Phi^0: \mathcal{H} \times \mathcal{U}_{ad} \to \mathcal{H} \times \mathcal{U}_{ad}$ as follows:

$$(z(t), \beta(t)) = \Phi^0(x,u)(t)$$

where

$$z(t) = S(t)x_0 + \int_0^{v(t)} (S(t-s)B_1 + S(t-r(r))B_2 r'(s))\beta(s)ds + \int_{v(t)}^t S(t-s)B_1\beta(s)ds + \int_0^t S(t-s)F(x,u)(s)ds + \int_0^t S(t-s)H(x,u)(s)dw(s),$$

where

$$u(t) = B_1^* S^*(T-t)\mathbb{E}\{(\Pi_0^T)^{-1}p(x)|\ \mathcal{F}_t\}, \qquad t \in [0, v(T)],$$

with

$$p(x) = x_T - S(T)x_0 - \int_0^{v(T)} S(T-s)F(x,u)(s)ds - \int_0^{v(T)} S(T-s)H(x,u)(s)dw(s),$$

$$u(t) = (B_1^* S^*(T-t) + B_2^* S^*(T-r(t)r'(t))\mathbb{E}\{(\Pi_0^T)^{-1}p(x)|\ \mathcal{F}_t\},\ t \in [v(T), T],$$

and with

$$p(x) = x_T - S(T)x_0 - \int_{v(T)}^T S(T-r(s))F(x,u)(s)ds - \int_{v(T)}^T S(T-r(s))H(x,u)(s)dw(s).$$

Now, we are ready to prove the relative exact controllability of (1).

Theorem 3 *Assume that Hypothesis 1 holds and the linear stochastic system (5) is relatively exactly controllable. Then the operator Φ^0 has a unique fixed point.*

Proof. The proof is similar to the proof of Theorem 1.

Theorem 4 *Assume that Hypothesis 1 is true and the linear stochastic system (5) is relatively exactly controllable on $[0,T]$, then the nonlinear stochastic system (1) is relatively exactly controllable.*

Proof. By Theorem 3, there exists a unique fixed point of an operator Φ^0. Let $(x^0,u^0)(\cdot)$ be the unique fixed point of an operator Φ^0. Then $x_T^0 = x_T$ for arbitrary $x_T \in \mathcal{L}_2(\mathcal{F}_T, \mathcal{H})$. Thus system (1) is relatively exactly controllable on $[0,T]$.

Corollary 2 *Assume that Hypothesis 1 is true. If the deterministic linear system corresponding to a linear stochastic system (5) is relatively exactly controllable on all $[0,t]$, $t > 0$, then nonlinear stochastic system (1) is relatively exactly controllable on $[0,T]$.*

Proof. By Theorem 4.2 from [9], the linear stochastic system (5) is relatively exactly controllable on $[0,T]$ if and only if the deterministic linear system corresponding to (5) small time relatively exactly controllable on $[0,T]$, that is, relatively exactly controllable on all $[0,t]$, $t > 0$. Then by Theorem 4 nonlinear stochastic system is relatively exactly controllable on $[0,T]$.

IV. Examples

Example 1 *Consider the following parabolic stochastic partial differential equation:*

$$\begin{aligned} dy &= [y_{\xi\xi} + B_1 u(t,\xi) + B_2 u(v(t),\xi) + f(t, y(t,\xi), \textstyle\int_0^t g(s,u(s,\xi))ds)]dt \\ &\quad + h(t, y(t,\xi), \textstyle\int_0^t k(s,u(s,\xi))ds)dw(s),\ t \in [0,T],\ \xi \in [0,\pi], \qquad (12) \\ y(t,0) &= y(t,\pi) = 0,\ t > 0. \end{aligned}$$

Let $\mathcal{H} = \mathcal{K} = L_2([0,\pi])$. Define $Ay = y''$ with

$$D(A) = \{y \in \mathcal{H} |\ y,\ y_\xi \text{ areabsolutelycontinuous}, y_{\xi\xi} \in \mathcal{H},\ y(0) = 0,\ y(\pi) = 0\},$$

then

$$Ay = \sum_{n=1}^{\infty} (-n^2)(y,\ e_n(\eta))e_n(\eta),\ y \in D(A)$$

with $e_n(\eta) = \sqrt{2/\pi}\sin n\eta, n = 1,\ 2,\ 3,\ \cdots, e_0 = 1$.

It is well known that A generates a strongly continuous semigroup $S(t), t > 0$.

Define an infinite dimensional space

$$\mathcal{U} = \{u = \sum_{n=2}^{\infty} u_n e_n \text{ with} \sum_{n=2}^{\infty} u_n^2 < \infty\}$$

with norm $\| u \| = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$.

Define a linear continuous mapping B_1 from $\mathcal{U}$ to $\mathcal{H}$ as follows

$$B_1 u = 2u_2 e_1(\eta) + \sum_{n=2}^{\infty} u_n e_n(\eta).$$

The system (12) can be written in abstract form given by (1) with $B_2 = I$. Then relative approximate controllability of the linear system associated with (12) follows from Theorem 5.2 [13]. In addition, if Hypothesis 1 is true. Then relative approximate controllability of (12) follows from Theorem 3.

Example 2 *Consider the following hyperbolic stochastic partial differential equation:*

$$\begin{aligned} d(\partial/\partial t)y = & \ [y_{\xi\xi} + B_1 u(t,\xi) + B_2 u(v(t),\xi) + f(t,y,\int_0^t g(s,u(s,\xi)))ds]dt \\ & +h(t,y,\int_0^t k(s,u(s,\xi))ds)dw(s),\ t \in [0,T],\ \xi \in [0,\pi], \\ y(t,0) = & \ y(t,1) = 0,\ t > 0, \\ y(0,\xi) = & \ \mu(\xi),\ (\partial/\partial t)y(0,\xi) = \nu(\xi). \end{aligned} \tag{13}$$

Let $\mathcal{H} = D(A^{1/2}) \oplus L_2([0,1])$, endowed with the inner product

$$\langle w,v \rangle = \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = \sum_{n=1}^{\infty} \{n^2\pi^2 \langle w_1,e_n\rangle\langle e_n,v_1\rangle + \langle w_2,e_n\rangle\langle e_n,v_2\rangle\}.$$

where $e_n(\eta) = \sqrt{2}\sin n\pi\eta, n = 1,\ 2,\ 3,\ \cdots$.

Let

$$z = \begin{bmatrix} y \\ (\partial/\partial t)y \end{bmatrix},\ z(0) = \begin{bmatrix} \mu \\ \nu \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$G = \begin{bmatrix} 0 \\ h \end{bmatrix},\ F = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Define $A_0 y = (d^2/d\xi^2)y$ and

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix},$$

with

$$D(A_0) = \{y \in L_2([0,1]) | y, y_\xi \text{ are absolutely continuous}, y_{\xi\xi} \in \mathcal{H}, y(0) = 0, y(1) = 0\}.$$

then system (13) can be written as

$$dz = (Az + B_1 u + B_2 u(v) + F(z,u))dt + G(z,u)dw,\ z(0) = \begin{bmatrix} \mu \\ \nu \end{bmatrix}. \quad (14)$$

It is well known that A is the infinitesimal generator of a contraction semigroup $S(t)$, $t > 0$.

It is well known that the linear stochastic system associated with (14) is relatively exactly controllable. In addition, if Hypothesis 1 is true. Then relative exact controllability (14) and hence that of (13) follows from Theorem 4.

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