Review of ReseaRch

ISSN: 2249-894X

Impact Factor : 5.7631(UIF)

Volume - 12 | Issue - 10 | July - 2023

A DISCUSSION ON METHODS OBTAINING LOWER AND UPPER BOUNDS OF STATISTICAL QUANTITIES BASED ON PROBABILITY MASS FUNCTION OF DISCRETE PROBABILITY DISTRIBUTION

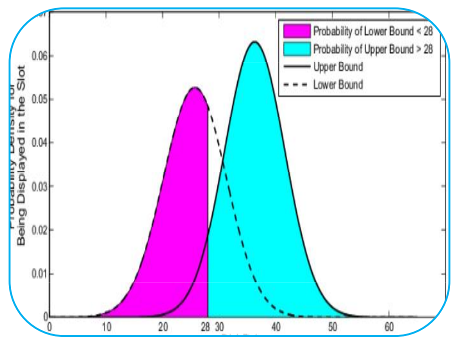
D. N. Kandekar[1] and R. B. Sonawane[2]

[1]Department of Mathematics, Dadapatil Rajale Arts, Science & Commerce College, Adinathnagar, Tal. :- Pathardi, Dist. :- Ahmednagar. (M.S.), India.

[2]Department of Mathematics, Ness Wadia College of Commerce Pune (M.S.), India.

ABSTRACT

Uncertainty plays an important role in our daily life. The belief function, commonality function, and plausibility function deal with uncertainty. The Belief function and the plausibility function give one of the important pairs of lower and upper bounds of the probability function respectively. From [8], we have another basic belief assignment based on probability mass function which is different from Shafer's basic belief assignment [12]. From this basic belief assignment, the belief function, commonality function, and plausibility function are obtained. In this paper, it is used to obtain lower and upper bounds of statistical quantities viz. distribution function, mean, variance, standard deviation, raw moments, central moments, coefficient of skewness, and coefficient of kurtosis of probability distribution under study with two methods. With help of algorithms, computer programs are written or constructed. Also, we obtain algorithms to obtain lower and upper bounds of statistical quantities of probability distribution under study. These bounds consist of probabilities given by the probability mass function of a probability distribution.



KEYWORDS: *Uncertainty, Belief function, Plausibility function, probability, probability mass function.*

Mathematics Subject Classification: 03B42.

INTRODUCTION

In the beginning, a special case of upper and lower probabilities has been introduced by Dempster [3, 4]. The existence of a probability function is assumed to be a mapping m from space X, where X is the frame of discernment Θ. The lower probability of A in X is equal to the probability of the largest subset of Θ such that its image under m is included in A. The upper probability of A in space X is that the image under m of all elements has a non-empty intersection with A. In [13], belief functions on a

[1] kandekardn@gmail.com

[2] sonawaneramdas@gmail.com

system of sets of an infinite or finite universe are represented by a probability measure or probability charge. In Kyburg's article [10], let set Π of all those probability distributions compatible with the available information

$$\forall \quad A \subseteq \Theta, P^*(A) = sup_{p\in\Pi}\, p(A) \quad P_*(A) = sup_{p\in\Pi}\, p(A)$$

with Π is a convex set of probability distributions. For confidence bands, $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$, where $F(x)$ is not precisely known and we can specify $\underline{F}(x)$ and $\overline{F}(x)$ from R to $[0,1]$. Then the distribution band is $\Gamma(\underline{F},\overline{F}) = (F \mid \forall \quad x \in \mathrm{R}, \underline{F}((-\infty,x]) \leq F(x) \leq \overline{F}(x))$ [9]. If $\underline{F}(x)$ and $\overline{F}(x)$ are step functions then distribution band s becomes probability box [5].

In [2], Imprecise belief structures are set of belief structures whose masses on focal elements A, interval-valued constraints $M = \{m : a_i \leq m(A_i) \leq b_i\}$. The intervals $[a_i b_i]$ specifying an Imprecise belief structures are not unique if $m(A_i) \leq min.\{b_i, 1-\sum_{j\neq i} a_j\}$. The upper and lower bounds to m determine interval ranges for belief and plausibility functions. In [14], Yager considers the same situations in which the masses of focal elements lie in some known interval, allowing us to model realistically situations in which the basic probability assignments can not be precisely identified.

In this paper, firstly we give preliminaries about discrete belief function theory, probability theory, and interval arithmetics. In the third section, we obtain the lower and upper bounds of the distribution function. In the fourth section, we explain the first method and derive a formula to obtain the lower and upper bound of statistical quantities of probability distribution under study. In the fifth section, we explain the second method and derive a formula to obtain the lower and upper bound of statistical quantities of probability distribution under study. In the sixth section, we provide algorithms to calculate the lower and upper bounds of statistical quantities of probability distribution under study. Finally, we explain the first and second methods by an illustrative example.

2. Preliminaries

Here we will provide necessary preliminaries about discrete belief function theory [12], interval arithmetics [11], and discrete distribution theory [1].

2.1 Discrete Belief Function Theory

Frame of Discernment :

Dictionary meaning of Frame of Discernment is the frame of good judgment insight. The word discerns means to recognize or find out or hear with difficulty. If the frame of discernment Θ is

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\} \tag{1}$$

then every element of Θ is a proposition. The set of all propositions of interest has a one-to-one correspondence with the set of all subsets of Θ, denoted by 2^Θ. A function $m : 2^\Theta \to [0,1]$ is called **basic probability assignment** whenever

$$\mathrm{m}(\varnothing) = 0 \text{ and } \sum_{A\subset\Theta} \mathrm{m}(A) = 1. \tag{2}$$

The quantity $m(A)$ is called A's **basic probability number** and it is a measure of the belief committed exactly to A. The *total belief* committed to A is the sum of $m(B)$, for all proper subsets B of A.

$$Bel(A) = \sum_{B \subset A} m(B). \tag{3}$$

If Θ is a frame of discernment, then a function $Bel: 2^{\Theta} \to [0,1]$ is called **belief function** over Θ if it satisfies above condition (3). A function $Bel: 2^{\Theta} \to [0,1]$ is *belief function* if and only if it satisfies following conditions

1. $Bel(\varnothing) = 0$.
2. $Bel(\Theta) = 1$.
3. For every positive integer n and every collection $A_1, A_2, \ldots, A_n$ of subsets of Θ

$$Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{I \subset \{1,2,\cdots,n\}} (-1)^{|I|+1} Bel(\bigcap_{i \in I} A_i). \tag{4}$$

Degree of doubt :

$$\mathrm{Dou(A) = Bel(\overline{A}) \text{ or } Bel(A) = 1 - Dou(\overline{A}).} \tag{5}$$

The quantity $pl(A) = 1 - Dou(A) = \sum_{A \cap B \neq \varnothing} m(B)$ which expresses the extent to which one finds A credible or plausible. We have a relation between the belief function, probability mass (or density) function, and plausibility function as:

$$Bel(A) \leq p(A) \leq Pl(A), \quad \forall A \subset \Theta. \tag{6}$$

If $|\Theta| = n$ then every element in the frame of discernment Θ is repeated exactly 2^{n-1} number of times and the sum of probabilities of all subsets of Θ is 2^{n-1}. Now, let $A = \{\{a_1\}, \{a_2\}, \ldots, \{a_n\}\} \subseteq \Theta$. In discrete space, since singletons are disjoint, the intersection of any number of singleton subsets of Θ is always an empty set. Therefore we have basic probability assignment [8] as:

$$m(A) = \frac{p(A)}{2^{n-1}}, \forall A \subseteq \Theta. \tag{7}$$

2.2 Interval Arithmetics

Interval arithmetic operations are useful to do calculations with intervals. We have interval arithmetic [11] as:

Let $X = [\underline{X}, \overline{X}]$ and $Y = [\underline{Y}, \overline{Y}]$ be any intervals, in set of real numbers. Here $\underline{X} = inf\{x : x \in X\}$ and $\overline{X} = sup\{x : x \in X\}$. Therefore $\underline{X}$ and $\overline{X}$ are lower and upper bounds of X respectively. The computations with intervals are as:

$$\begin{aligned} X + Y &= [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]. \\ X - Y &= [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}]. \\ X \cdot Y &= [Min.S, Max.S], \quad \text{where} S = \{\underline{X}\,\underline{Y}, \underline{X}\,\overline{Y}, \overline{X}\,\underline{Y}, \overline{X}\,\overline{Y}\}. \end{aligned} \tag{8}$$

$$X/Y = \begin{cases} [\underline{X}, \overline{X}][1/\overline{Y}, 1/\underline{Y}] & \text{if } 0 \notin [\underline{Y}, \overline{Y}] \\ [-\infty, \infty] & \text{if } 0 \in [\underline{X}, \overline{X}] \text{ and } 0 \in [\underline{Y}, \overline{Y}] \\ [X/Y, \infty] & \text{if } \overline{X} \leq 0 \text{ and } \overline{Y} = 0 \\ [-\infty, X/Y] \cup [X/Y, \infty] & \text{if } \overline{X} \leq 0 \text{ and } \underline{Y} < 0 < \overline{Y} \\ [-\infty, X/Y] & \text{if } \overline{X} \leq 0 \text{ and } \underline{Y} = 0 \\ [-\infty, \infty] & \text{if } \underline{X} < 0 < \overline{X} \text{ and } \underline{Y} < 0 < \overline{Y} \\ [-\infty, X/Y] & \text{if } \underline{X} \geq 0 \text{ and } \overline{Y} = 0 \\ [-\infty, X/Y] \cup [\underline{X}/\overline{Y}, \infty] & \text{if } \underline{X} \geq 0 \text{ and } \underline{Y} < 0 < \overline{Y} \\ [X/Y, \infty] & \text{if } \underline{X} \geq 0 \text{ and } \underline{Y} = 0 \\ \varnothing & \text{if } 0 \notin [\underline{X} < 0 < \overline{X}] \text{ and } \overline{Y} = 0 \end{cases} \tag{9}$$

$$\begin{aligned} & f(X) == \{f(x) \mid x \in X\} \quad . \\ & |X| = \max.\{\underline{X}, \overline{X}\} \quad . \\ & X^n = \{x^n \mid x \in X\} = \begin{cases} [\underline{X}^n, \overline{X}^n], & \text{if } \underline{X} > 0 \text{ or n is odd} \\ [\overline{X}^n, \underline{X}^n] & \text{if } \overline{X} < 0 \text{ and n is even.} \\ [0, |X|^n] & \text{if } 0 \in X \text{ and n is even} \end{cases} \end{aligned} \tag{10}$$

2.3 Indexing of subsets of Θ and Some Statistical Quantities

To apply statistical concepts for our defined basic belief assignment, we apply indexing of subsets of Θ as follows:

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ hence $|\Theta| = n$. The number of subsets of Θ is 2^n. We define indicator function as :

$$\text{For any subset A of } \Theta, I_A(\theta_i) = \begin{cases} 0 & \text{if } \theta_i \notin A \\ 1 & \text{if } \theta_i \in A. \end{cases} \tag{11}$$

If $A = \{\theta_j, \theta_k, \theta_l, \theta_m, \theta_p, \theta_q\}$ then indexing number of A in Θ is

$$v = \sum_{i=1}^{n} I_A(\theta_i) 2^{i-1} = 2^{j-1} + 2^{k-1} + 2^{l-1} + 2^{m-1} + 2^{p-1} + 2^{q-1} \tag{12}$$

Notes :-

1. $0 \leq v \leq 2^n - 1$.
2. $v = 0$ corresponds to $\varnothing$.
3. $v = 2^n - 1$ corresponds to Θ.
4. Any value in between 0 and $2^n - 1$ corresponds to proper subset of Θ.
5. Indexing of subsets of Θ helps in obtaining statistical quantities as it does not affect the results of statistics and mathematics.

With this indexing of set, we obtain some statistical quantities [1] as :

1. Distribution Function : $P(x) = P[X \leq x]$.
2. Expectation of V = Mean: $E(V) = \sum_{V=0}^{2^n-1} Vp(V)$.
3. r^{th} raw moment : $\mu_r' = E(V^r) = \sum_{V=0}^{2^n-1} V^r p(V)$.
4. r^{th} central moment : $\mu_r = E((V - E(V))^r) = \sum_{V=0}^{2^n-1} (V - E(V))^r p(V)$.
5. Variance : $Var(V) = E(V^2) - (E(V))^2$.
6. Standard Deviation : $\sigma_V = \sqrt{Var(V)}$.
7. Coefficient of Skewness : $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$.
8. Coefficient of Kurtosis : $\beta_2 = \frac{\mu_4}{\mu_2^2}$.

3 The Lower and Upper Bounds of Distribution Function

By indexing of subsets of Θ, we have

X	X_0	X_1	X_2	X_3	X_4
V	*1*	*2*	*4*	*8*	*16*

Now subsets of Θ required for the probability distribution, have a relation between X and V as shown in Table 1.

Sr. No.	**Subset of Θ**	V
1	$\{x_0\} = \{0\}$	1
2	$\{x_0, x_1\} = \{0,1\}$	1+2=3
3	$\{x_0, x_1, x_2\} = \{0,1,2\}$	1+2+4=7
4	$\{x_0, x_1, x_2, x_3\} = \{0,1,2,3\}$	1+2+4+8=15
5	$\{x_0, x_1, x_2, x_3, x_4\} = \{0,1,2,3,4\}$	1+2+4+8+16=31
	$\vdots$	$\vdots$
k	$\{x_0, x_1, x_2, \ldots, x_k\} = \{0,1,2,\ldots,k\}$	$1+2+4+\cdots+2^k = 2^{k+1}-1$

Table 1: Indexing of Subsets of Θ

Now by notation

$$p(v) = p(A_v) \quad v = 0,1,2,3,\cdots,2^n - 1$$

By indexing of sets, $F(x) = P(X \leq x) = p(\{0,1,2,3,\ldots,x\}) = p(A_v)$ and only in this case, relation between x and v is $v = 2^{x+1} - 1, \quad x = 0,1,2,3,\ldots,n$. By lower and upper bounds of the probability of sets (6), $Bel(A_v) \leq P(A_v) \leq Pl(A_v)$, we get $Bel(A_v) \leq F(X) \leq x) \leq Pl(v), \quad x = 0,1,2,3,\ldots,n$ and $v = 2^{x+1} - 1$ Therefore we get lower and upper bounds of the distribution function of given probability distribution including the case of a subset $\varnothing$.

4 First Method

If $p(A)=0$ then $Bel(A)=0$ and $Pl(A)=0$ hence $Bel(A)=p(A)=Pl(A)=0$. We have, for any subset $A\subseteq\Theta$ with $p(A)\neq 0$, by using series results [6], we have

$$\begin{aligned}
&Bel(A)\le p(A)\le Pl(A), \quad \forall A\subset\Theta.\\
&\Rightarrow \frac{Bel(A)}{p(A)}\le 1\le \frac{Pl(A)}{p(A)} \text{provided } p(A)\neq 0.\\
&\Rightarrow \sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)}\le \sum_{A\subseteq\Theta}1\le \sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)}\\
&\Rightarrow \sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)}\le \sum_{v=1}^{2^n-1}1\le \sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)}\\
&\Rightarrow \sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)}\le 2^n-1\le \sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)}\\
&\Rightarrow \frac{1}{2^n-1}\sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)}\le 1\le \frac{1}{2^n-1}\sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)}\\
&\Rightarrow \frac{1}{2^n-1}\sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)}(Stat.Quant.)_{pd}\le (Stat.Quant.)_{pd}\le \frac{1}{2^n-1}\sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)}(Stat.Quant.)_{pd}.
\end{aligned} \tag{13}$$

where $(Stat.Quant.)_{pd}$ is statistical quantity based on probability distribution under study. Note that we discard quantities $\frac{Bel(A)}{p(A)}$ and $\frac{Pl(A)}{p(A)}$ where $p(A)=0$. This is important in obtaining approximate lower and upper bounds of statistical quantities.

4.1 Proper Magnification or Reduction of Upper and Lower bounds of Raw Moments

The above formula is applicable for the calculation of statistical quantities viz. raw moments only and we obtain lower and upper bounds of r^{th} raw moments of probability distribution under study as interval $[\underline{\mu}'_r, \overline{\mu}'_r]$, where lower bound of interval represents the lower bound of r^{th} raw moment and upper bound of interval represents the upper bound of r^{th} raw moment.

$$\begin{aligned}
\frac{1}{2^n-1}\sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)}\mu'_r &\le \mu'_r \le \frac{1}{2^n-1}\sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)}\mu'_r\\
\underline{\mu}'_r &\le \mu'_r \le \overline{\mu}'_r
\end{aligned} \tag{14}$$

where $\mu'_r = \text{corresponding}\, r^{th}$ raw moment of concerned probability distribution [1, 2].

4.2 Central Moments

In [1], we have central moments of probability distribution as:

$$\begin{aligned}
\mu_1 &= 0\\
\mu_2 &= \mu'_2-(\mu'_1)^2\\
\mu_3 &= \mu'_3-3\mu'_2\mu'_1+2(\mu'_1)^3\\
\mu_4 &= \mu'_4-4\mu'_3\mu'_1+6\mu'_2\mu_1'^2-3(\mu'_1)^4
\end{aligned} \tag{15}$$

Lower and Upper Bounds of Central Moments :-

Using intervals for raw moments (14), formulae for central moments based on raw moments (15), and rules of interval arithmetics (8), (9) and (10), we obtain lower and upper bounds for central moments of probability distribution under study. Here $p(A) \geq 0$ hence $Bel(A) \geq 0, \quad Pl(A) \geq 0 \text{and} 1/p(A) \geq 0$. The quantities $\sum_{A \subseteq \Theta} \frac{Bel(A)}{p(A)} \geq 0$ and $\sum_{A \subseteq \Theta} \frac{Pl(A)}{p(A)} \geq 0$. Therefore, the quantities $\frac{1}{2^n - 1} \sum_{A \subseteq \Theta} \frac{Bel(A)}{p(A)} \geq 0$ and $\frac{1}{2^n - 1} \sum_{A \subseteq \Theta} \frac{Pl(A)}{p(A)} \geq 0$. The formulae for central moments in terms of intervals become as: For the first central moment, we have

$$\begin{aligned} \mu_1 &= 0 \\ &= \mu_1' - \mu_1' \\ &= [\underline{\mu}_1', \overline{\mu}_1'] - [\underline{\mu}_1', \overline{\mu}_1'] \\ &= [\underline{\mu}_1' - \overline{\mu}_1', \overline{\mu}_1' - \underline{\mu}_1']. \end{aligned} \tag{16}$$

For the second central moment, we have

$$\begin{aligned} \mu_2 &= \mu_2' - \mu_1'^2 \\ &= [\underline{\mu}_2', \overline{\mu}_2'] - [\underline{\mu}_1', \overline{\mu}_1']^2 \\ &= [\underline{\mu}_2' - \overline{\mu}_1'^2, \overline{\mu}_2' - \underline{\mu}_1'^2]. \end{aligned} \tag{17}$$

For the third central moment, we have

$$\begin{aligned} \mu_3 &= \mu_3' - 3(\mu_2')(\mu_1') + 2\mu_1'^2 \\ &= [\underline{\mu}_3', \overline{\mu}_3'] - 3[\underline{\mu}_2', \overline{\mu}_2'][\underline{\mu}_1', \overline{\mu}_1'] + 2[\underline{\mu}_1', \overline{\mu}_1']^3 \\ &= [\underline{\mu}_3' - 3(\overline{\mu}_2')(\overline{\mu}_1') + 2\underline{\mu}_1'^3, \overline{\mu}_3' - 3(\underline{\mu}_2')(\underline{\mu}_1') + 2\overline{\mu}_1'^3]. \end{aligned} \tag{18}$$

For the fourth central moment, we have

$$\begin{aligned} \mu_4 &= \mu_4' - 4(\mu_3')(\mu_1') + 6(\mu_2')(\mu_1')^2 - 3\mu_1'^4 \\ &= [\underline{\mu}_4', \overline{\mu}_4'] - 4[\underline{\mu}_3', \overline{\mu}_3'][\underline{\mu}_1', \overline{\mu}_1'] \\ &+ 6[\underline{\mu}_2', \overline{\mu}_2'][\underline{\mu}_1', \overline{\mu}_1']^2 - 3[\underline{\mu}_1', \overline{\mu}_1']^4 \\ &= [\underline{\mu}_4' - 4\overline{\mu}_3'\overline{\mu}_1' + 6\underline{\mu}_2'\underline{\mu}_1'^2 - 3\overline{\mu}_1'^4, \\ &\overline{\mu}_4' - 4\underline{\mu}_3'\underline{\mu}_1' + 6\overline{\mu}_2'\overline{\mu}_1'^2 - 3\underline{\mu}_1'^4]. \end{aligned} \tag{19}$$

4.3 Coefficients of Skewness and Kurtosis

Using interval arithmetic (8), (9), and (10) and lower and upper bounds of central moments (17), (18), and (19), we obtain the lower and upper bounds of coefficient of skewness and coefficient of kurtosis as:

$$\begin{aligned}
\text{Coefficient of Skewness} = \beta_1 &= \frac{\mu_3^{\,2}}{\mu_2^{\,3}} \\
&= \frac{[\underline{\mu}_3, \overline{\mu}_3]^2}{[\underline{\mu}_2, \overline{\mu}_2]^3} \\
&= \frac{[0, \overline{\mu}_3^{\,2}]}{[\underline{\mu}_2^{\,3}, \overline{\mu}_2^{\,3}]} \\
\text{Coefficient of Kurtosis} = \beta_2 &= \frac{\mu_4}{\mu_2^{\,2}} \\
&= \frac{[\underline{\mu}_4, \overline{\mu}_4]}{[\underline{\mu}_2, \overline{\mu}_2]^2} \\
&= \frac{[\underline{\mu}_4, \overline{\mu}_4]}{[0, \overline{\mu}_2^{\,2}]}.
\end{aligned} \tag{20}$$

By using series results [6], consider

$$\begin{aligned}
\sum_{A\subseteq\Theta} \frac{Bel(A)}{p(A)} &= \sum_{A\subseteq\Theta} Bel(A)\frac{1}{p(A)} \\
&= \sum_{A\subseteq\Theta} \frac{p(A)}{2^{n-k}}\frac{1}{p(A)} \quad \text{where k is cardinality of } A\subseteq\Theta \\
&= \sum_{A\subseteq\Theta} \frac{1}{2^{n-k}} \\
&= \sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^{n-k}} \\
&= \sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^{n}\cdot 2^{-k}} \\
&= \frac{1}{2^n}\sum_{k=1}^{n} 2^k \binom{n}{k} \\
&= \frac{1}{2^n}\sum_{k=0}^{n} 2^k \binom{n}{k} - \frac{1}{2^n} \\
&= \frac{1}{2^n}(1+2)^n - \frac{1}{2^n}, \text{ where } \sum_{k=0}^{n} x^k \binom{n}{k} = (1+x)^n \\
&= \frac{1}{2^n}3^n - \frac{1}{2^n} \\
&= \frac{3^n}{2} - \frac{1}{2^n}.
\end{aligned}$$

Therefore

$$\frac{1}{2^n-1}\sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)} = \frac{1}{2^n-1}\{\frac{3^n}{2}-\frac{1}{2^n}\} = \frac{3^n-1}{2^n(2^n-1)}. \tag{21}$$

Now, consider

$$\sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)} = \sum_{A\subseteq\Theta}\frac{p(A)+\frac{2^k-1}{2^k}p(\bar{A})}{p(A)}$$

where, $Pl(A) = p(A)+\frac{2^k-1}{2^k}p(\bar{A})$, with $k = |A|$ and $\bar{A}$ is complement of $A \subseteq \Theta$.

$$\begin{aligned}\sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)} &= \sum_{A\subseteq\Theta}1+\frac{2^k-1}{2^k}\frac{p(\bar{A})}{p(A)} \\ &= \sum_{A\subseteq\Theta}1+\sum_{A\subseteq\Theta}\frac{2^k-1}{2^k}\frac{p(\bar{A})}{p(A)} \\ &= \sum_{v=1}^{2^n-1}1+\sum_{A\subseteq\Theta}\frac{2^k-1}{2^k}\frac{p(\bar{A})}{p(A)} \\ &= (2^n-1)+\sum_{A\subseteq\Theta}\frac{2^k-1}{2^k}\frac{p(\bar{A})}{p(A)} \\ &= (2^n-1)+\sum_{A\subseteq\Theta}\frac{2^k-1}{2^k}\frac{1-p(A)}{p(A)} \\ &= (2^n-1)+\sum_{A\subseteq\Theta}\frac{2^k-1}{2^k}(\frac{1}{p(A)}-1), \text{provided } p(A)\neq 0 \\ &= (2^n-1)+\sum_{A\subseteq\Theta}\frac{2^k-1}{2^k}\frac{1}{p(A)}-\sum_{A\subseteq\Theta}\frac{2^k-1}{2^k}.\end{aligned} \tag{22}$$

Since $0\leq p(A)\leq 1, \forall A\subseteq\Theta$, the quantity $\frac{1}{p(A)}$ lies between 1 and ∞. Therefore it becomes difficult to calculate $\sum_{A\subseteq\Theta}\frac{Pl(A)}{p(A)}$ hence the upper bound of the statistical quantity of probability distribution under study but this upper bound is less than ∞ as $p(A)\neq 0 \text{ for some } A\subseteq\Theta$.

Remark: Consider, for any $A\subseteq\Theta$,

$$\frac{Bel(A)}{p(A)} = \frac{2^{|A|-1}m(A)}{p(A)} = \frac{2^{|A|-1}\frac{p(A)}{2^{n-1}}}{p(A)} = \frac{2^{|A|-1}}{2^{n-1}}.$$

Therefore, for any $A\subseteq\Theta$,

$$\begin{aligned}
\sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)} &= \sum_{A\subseteq\Theta}\frac{2^{|A|-1}}{2^{n-1}} \\
&= \frac{1}{2^{n-1}}\sum_{A\subseteq\Theta}2^{|A|-1}2^{n-1} \\
&= \frac{1}{2^{n-1}}\sum_{k=1}^{n}\binom{n}{k}2^{k-1} \\
&= \frac{1}{2^{n}}\sum_{k=1}^{n}\binom{n}{k}2^{k} \\
&= \frac{1}{2^{n}}\sum_{k=0}^{n}\binom{n}{k}2^{k}-\frac{1}{2^{n}} \\
&= \frac{(1+2)^{n}}{2^{n}}-\frac{1}{2^{n}} \\
&= \frac{3^{n}}{2^{n}}-\frac{1}{2^{n}} \\
&= \frac{3^{n}-1}{2^{n}}.
\end{aligned} \tag{23}$$

If for some $A\subseteq\Theta,\quad p(A)=0$ then substract corresponding term $\frac{1}{2^{n-|A|}}$ from $\sum_{A\subseteq\Theta}\frac{Bel(A)}{p(A)}=\frac{3^{n}-1}{2^{n}}$.

Repeat this step for all $A\subseteq\Theta,\quad \text{with}\, p(A)=0$. Therefore from the equations, we have

$$\frac{3^{n}-1}{2^{n}(2^{n}-1)}(Stat.Quant.)_{pd}\leq(Stat.Quant.)_{pd}\leq\frac{1}{2^{n}-1}\{(2^{n}-1)+\sum_{A\subseteq\Theta}\frac{2^{k}-1}{2^{k}}\frac{1}{p(A)}-\sum_{A\subseteq\Theta}\frac{2^{k}-1}{2^{k}}\}(Stat.Quant.)_{pd}$$

5 Second Method

From [8], we have

$$\begin{aligned}
\sum_{A\subseteq\Theta}Bel(A) &= \left(\frac{3}{2}\right)^{n-1}, \\
\sum_{A\subseteq\Theta}p(A) &= 2^{n-1}, \\
\text{and}\sum_{A\subseteq\Theta}Pl(A) &= 2^{n-1}+\sum_{r=1}^{n-1}\binom{n-1}{r}(1-2^{-r}).
\end{aligned} \tag{24}$$

Also, we have, for any $A\subseteq\Theta$,

$$Bel(A) \le p(A) \le Pl(A)$$
$$\Rightarrow \sum_{A\subseteq\Theta} Bel(A) \le \sum_{A\subseteq\Theta} p(A) \le \sum_{A\subseteq\Theta} Pl(A)$$
$$\Rightarrow (\frac{3}{2})^{n-1} \le 2^{n-1} \le 2^{n-1} + \sum_{r=1}^{n-1}\binom{n-1}{r}(1-2^{-r})$$
$$\Rightarrow \frac{(\frac{3}{2})^{n-1}}{2^{n-1}} \le 1 \le \frac{2^{n-1} + \sum_{r=1}^{n-1}\binom{n-1}{r}(1-2^{-r})}{2^{n-1}}$$
$$\Rightarrow \frac{3^{n-1}}{2^{2n-2}} \le 1 \le 1 + \frac{1}{2^{n-1}}\sum_{r=1}^{n-1}\binom{n-1}{r}(1-2^{-r})$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + \frac{1}{2^{n-1}}\sum_{r=1}^{n-1}\binom{n-1}{r}(1-2^{-r})$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + \frac{1}{2^{n-1}}(\sum_{r=1}^{n-1}\binom{n-1}{r} - \sum_{r=1}^{n-1}\binom{n-1}{r}2^{-r})$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + \frac{1}{2^{n-1}}(\sum_{r=1}^{n-1}\binom{n-1}{r} - \sum_{r=1}^{n-1}\binom{n-1}{r}(\frac{1}{2})^{r})$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + \frac{1}{2^{n-1}}((2^{n-1}-1) - ((1+\frac{1}{2})^{n-1}-1))$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + \frac{1}{2^{n-1}}(2^{n-1}-1-(1+\frac{1}{2})^{n-1}+1)$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + \frac{1}{2^{n-1}}(2^{n-1}-(1+\frac{1}{2})^{n-1}$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + \frac{1}{2^{n-1}}(2^{n-1}-(\frac{3}{2})^{n-1})$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 1 + 1 - (\frac{3}{2})^{n-1}\frac{1}{2^{n-1}})$$
$$\Rightarrow (\frac{3}{4})^{n-1} \le 1 \le 2 - (\frac{3}{4})^{n-1}$$

$$\Rightarrow (\frac{3}{4})^{n-1} \cdot (Stat.Quant.)_{pd} \le (Stat.Quant.)_{pd} \le (2-(\frac{3}{4})^{n-1}) \cdot (Stat.Quant.)_{pd}. \quad (25)$$

The procedure to obtain lower and upper bounds of statistical quantities of probability distribution under study is similar to the procedure in the first Method with change in quantities BP and PlP. The quantities BP and PlP are replaced by $(\frac{3}{4})^{n-1}$ and $2-(\frac{3}{4})^{n-1}$ respectively.

6 Algorithms

Algorithms play an important role in the logical sequencing of tasks or operations in any procedure. It helps in framing or constructing or writing computer programs hence mechanizing the whole procedure. In this section, we obtain algorithms for indexing, probability and basic probability numbers, belief function, commonality function, and plausibility function of subsets of Θ and summation of belief function, commonality function, and plausibility function of subsets of Θ with ratios of belief function to probability function and plausibility function to probability function in the first method and the second method. Also, we obtain algorithms for the lower and upper bounds of raw moments and central moments. Here, the number of distinct elements of Θ is n, the number of subsets of Θ is $2^n - 1$, $P(\cdot)$ represent the probability of singleton subset $\{\theta_i\}$ of Θ, and $P_i, Bel_i, Pl_i, \text{and} Comm_i$ represent belief, plausibility, and commonality of the subset with index i of Θ respectively.

A: Indexing, Probability, and Basic Probability Numbers of Subsets of Θ:

Input n For $i = 0$ to $2^n - 1$

$V_i = 0$ and $P_i = 0$

For $j = 1$ to n

For $I_i(\theta_j) = 0$ to 1

$V_i = V_i + I_i(\theta_j) * 2^{i-1}$

$P_i = P_i + I_i(\theta_j) * P(\theta_j)$

next $I_i(\theta_j)$

next j

$m_i = \frac{P_i}{2^{n-1}}$

next i

B: Belief Function of Subsets of Θ:

For $i = 0$ to $2^n - 1$

$Bel_i = 0$

For $k = 0$ to $2^n - 1$

For $j = 1$ to n

if $I_k(\theta_j) \le I_i(\theta_j)$ then next j else next k

$Bel_i = Bel_i + m_k$

next k

next i

C: Plausibility Function of Subsets of Θ:

For $i = 0$ to $2^n - 1$

$Pl_i = 0$

For $k = 0$ to $2^n - 1$

For $j = 1$ to n

if $I_k(\theta_j) \ne I_i(\theta_j) = 1$ then next j else $Pl_i = Pl_i + m_k$ next k

next k

next i

D: Commonality Function of Subsets of Θ :

For $i = 0$ to $2^n - 1$

$Comm_i = 0$

For $k = 0$ to $2^n - 1$

For $j = 1$ to n

if $I_k(\theta_j) \geq I_i(\theta_j)$ then next j else next k

$Comm_i = Comm_i + m_k$

next k

next i

E: Summation of Belief Function, Commonality Function, and Plausibility Function of subsets of Θ :

Here Bel and Pl represent the summation of belief functions and plausibility functions of all subsets of Θ respectively.

$Bel = 0, Comm = 0, Pl = 0$

For $i = 0$ to $2^n - 1$

$Bel = Bel + Bel_i$

$Comm = Comm + Comm_i$

$Pl_i = Pl + Pl_i$

next i

6.1 First Method

Calculation of BP and PlP:

Here, BP and PlP represent the summation of ratios of belief and plausibility of a subset of Θ respectively with a probability of the same subset of Θ. while calculating BP and PlP, we consider those subsets A_i of frame of discernment Θ with $P(A_i) = P_i \neq 0$ hence neglect those subsets A_i of frame of discernment Θ with $P(A_i) = P_i = 0$.

$BP = 0, PlP = 0$

For $i = 1$ to $2^n - 1$
if $P_i \neq 0$ then

{

$BP = BP + \dfrac{Bel_i}{P_i}$

$PlP = PlP + \dfrac{Pl_i}{P_i}$

}

else next i

next i

6.2 Second Method

A: Calculation of BP and PlP:

Here, BP and PlP represent ratios of summation of belief and plausibility of a subset of Θ with probability same subset of Θ respectively.

$$\text{For } i = 1 \text{ to } 2^n - 1$$
$$Bel = Bel + Bel_i$$
$$P = P + P_i$$
$$Pl_i = Pl + Pl_i$$
$$\text{next} i$$
$$BP = \frac{Bel}{P},$$
$$PlP = \frac{Pl}{P}.$$

B: r^{th} raw moment = $\mu_r^{'}$: ,

Here $RML_i, RMU_i, \text{and} RM_i$ represent lower bound of i^{th} raw moment, upper bound of i^{th} raw moment, and i^{th} raw moment respectively.

$$\text{For } i = 1 \text{ to } 4$$
$$RML_i = BP * RM_i,$$
$$RMU_i = PIP * RM_i, \tag{26}$$
$$\text{next } i,$$

C: Calculation of Central Moments:

Here CML_i and CMU_i represent the lower bound of i^{th} central moment, and the upper bound of i^{th} central moment respectively.

$$\begin{aligned}
CML_1 &= RML_1 - RMU_1 \\
CMU_1 &= RMU_1 - RML_1 \\
CML_2 &= RNL_2 - RMU_2^{\ 2} \\
CMU_2 &= RMU_2 - RML_2^{\ 2} \\
CML_3 &= RML_3 - 3(RMU_2)(RMU_1) + 2RML_1^{\ 3} \\
CMU_3 &= RMU_3 - 3(RML_2)(RML_1) + 2RMU_1^{\ 3} \\
CML_4 &= RML_4 - 4(RMU_3)(RMU_1) + 6(RML_2)(RML_1)^2 - 3RMU_1^{\ 4} \\
CMU_4 &= RMU_4 - 4(RML_3)(RML_1) + 6(RMU_2)(RMU_1)^2 - 3RML_1^{\ 4}
\end{aligned} \tag{27}$$

These algorithms emphasize major steps of operations consisting of sequencing and looping hence these algorithms help frame or construct computer programs.

7. Illustrative Example

Now we illustrate both methods by an example. Let $X : Binomial(n, p)$. Therefore $p(x) = \binom{n}{p} p^x q^{n-x}$ [1]. Now connsider $n = 4, p = 2/3$ and $q = 1 - p = 1/3$. The distribution of X is

X	0	1	2	3	4	Total
p(x)	1/81	8/81	24/81	32/81	16/81	1

(28)

We have raw moments, central moments, coefficient of skewness, and coefficient of kurtosis of Binomial distribution under study as:

$$\mu_1' = \sum xp(x) = 216/81 = 2.66666667, \qquad \mu_2' = \sum x^2 p(x) = 648/81 = 8,$$

$$\mu_3' = \sum x^3 p(x) = 2088/81 = 25.77777778 \qquad \mu_4' = \sum x^4 p(x) = 7080/81 = 87.4074074$$

$$\mu_1 = 0, \qquad \mu_2 = \mu_2' - \mu_1'^2 = 0.8889,$$

$$\mu_3 = \mu_3' - 3\mu_2'^2 + 2\mu_1'^3 = -0.2963 \qquad \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 2.0741$$

And coefficient of skewness

$$= \beta_1 = \frac{\mu_3^2}{\mu_2^3} = -0.124998, \qquad \text{coefficientofkurtosis} \beta_2 = \frac{\mu_4}{\mu_2^2} = 2.624967.$$

7.1 First Method

Sr. No.	**subset of Θ**	$p(\cdot)$	$m(\cdot)$	$Bel(\cdot)$	$Pl(\cdot)$	$\frac{Bel(\cdot)}{p(\cdot)}$	$\frac{Pl(\cdot)}{p(\cdot)}$
1	$\varnothing$	0	0	0	0	—	—
2	{0}	1/81	1/1296	1/1296	41/81	81/1296=1/16	41
3	{1}	8/81	8/1296	8/1296	89//162	81/1296=1/16	89/16
4	{0, 1}	9/81	9/1296	18/1296	63//81	162/1296=2/16	7
5	{2}	24/81	24/1296	24/1296	105/162	81/1296=1/16	105/48
6	{0, 2}	25/81	25/1296	50/1296	67/81	162/1296=2/16	67/25
7	{1, 2}	32/81	32/1296	64/1296	275/324	162/1296=2/16	275/128
8	{0, 1, 2}	33/81	33/1296	132/1296	75/81	324/1296=4/16	75/33
9	{3}	32/81	32/1296	32/1296	113/162	81/1296=1/16	113/128
10	{0, 3}	33/81	33/1296	66/1296	69/81	162/1296=2/16	69/33
11	{1, 3}	40/81	40/1296	80/1296	283/324	162/1296=2/16	283/160
12	{0, 1, 3}	41/81	41/1296	164/1296	76/81	324/1296=4/16	76/41
13	{2, 3}	56/81	56/1296	112/1296	299/324	162/1296=2/16	299/224
14	{0, 2, 3}	57/81	57/1296	228/1296	78/81	324/1296=4/16	78/65

15	{1, 2, 3}	64/81	64/1296	256/1296	631/648	324/1296=4/16	631/512
16	{0, 1, 2, 3}	65/81	65/1296	520/1296	80/81	648/1296=8/16	80/65
17	{4}	16/81	16/1296	16/1296	97/162	81/1296=1/16	97/64
18	{0, 4}	17/81	17/1296	34/1296	65/81	162/1296=2/16	65/17
19	{1, 4}	24/81	24/1296	48/1296	267/324	162/1296=2/16	267/96
20	{0, 1, 4}	25/81	25/1296	100/1296	74/81	324/1296=4/16	74/25
21	{2, 4}	40/81	40/1296	80/1296	283/324	162/1296=2/16	283/160
22	{0, 2, 4}	41/81	41/1296	164/1296	76/81	324/1296=4/16	76/41
23	{1, 2, 4}	48/81	48/1296	192/1296	615/648	324/1296=4/16	615/384
24	{0, 1, 2, 4}	49/81	49/1296	392/1296	79/81	648/1296=8/16	79/49
25	{3, 4}	48/81	48/1296	96/1296	291/324	162/1296=2/16	291/192
26	{0, 3, 4}	49/81	49/1296	196/1296	77/81	324/1296=4/16	77/49
27	{1, 3, 4}	56/81	56/1296	224/1296	623/648	324/1296=4/16	623/448
28	{0, 1, 3, 4}	57/81	57/1296	456/1296	159/162	648/1296=8/16	159/114
29	{2, 3, 4}	72/81	72/1296	288/1296	639/648	324/1296=4/16	639/576
30	{0, 2, 3, 4}	73/81	73/1296	584/1296	161/162	648/1296=8/16	161/146
31	{1, 2, 3, 4}	80/81	80/1296	640/1296	1295/1296	648/1296=8/16	1295/1280
32	{0, 1, 2, 3, 4}	81/81=1	81/1296	1296/1296=1	81/81=1	1296/1296=16/16=1	1296/1296=1
$\sum$	**Total**	1296/81=16	1	6561/1296	26.5455247	121/16	102.458177

Table 2: Calculation of Belief Functions $Bel, Pl,$ and $Bel/P, Pl/P$

From the Table 2, we have

1. $\sum_{A\subseteq\Theta} p(A) = 2^{5-1} = 2^4 = 16 = 1296/81$.
2. $\sum_{A\subseteq\Theta} m(A) = 1$.
3. for any subset $A \subseteq \Theta$, $Bel(A) = m(A)\cdot 2^{|A|-1}$, where $|A| =$ cardinality of set A.
4. for any subset

$$A \subseteq \Theta, \quad Pl(A) = p(A) + \left(\frac{2^k - 1}{2^k}\right) p(\bar{A}), \quad \text{where } k = |A| \text{ and } \bar{A} = \text{complement of } A \text{ in } \Theta .$$

We have

$$\frac{1}{2^n - 1}\sum_{A\subseteq\Theta} \frac{Bel(A)}{p(A)} (Stat.Quant.)_{pd} \le (Stat.Quant.)_{pd} \le \frac{1}{2^n - 1}\sum_{A\subseteq\Theta} \frac{Pl(A)}{p(A)} (Stat.Quant.)_{pd} .$$

Here $n = 5$, and from Table 2, we have $\sum_{A\subseteq\Theta} \frac{Bel(A)}{p(A)} = 121/16, \quad \sum_{A\subseteq\Theta} \frac{Pl(A)}{p(A)} = 102.458177$.

Therefore the relationship becomes

$$\frac{1}{2^5 - 1}(121/16)(Stat.Quant.)_{pd} \le (Stat.Quant.)_{pd} \le \frac{1}{2^5 - 1}(102.458177)(Stat.Quant.)_{pd} .$$

$$\Rightarrow (0.243951613)(Stat.Quant.)_{pd} \leq (Stat.Quant.)_{pd} (3.30510248)(Stat.Quant.)_{pd}. \qquad (29)$$

Using the above equation (29), rules of operations on intervals (interval arithmetics) (8), (9), and (10), and raw moments of discrete binomial probability distribution under study, we have lower and upper bounds for raw moments as:

$$\begin{aligned}
\underline{\mu}_1^{'} &= 0.650537635 & \leq \mu_1^{'} \leq 8.81360662 = \overline{\mu}_1^{'} \\
\underline{\mu}_2^{'} &= 1.9516129 & \leq \mu_2^{'} \leq 26.4408198 = \overline{\mu}_2^{'} \\
\underline{\mu}_3^{'} &= 6.28853047 & \leq \mu_3^{'} \leq 85.1981973 = \overline{\mu}_3^{'} \\
\underline{\mu}_4^{'} &= 21.323178 & \leq \mu_4^{'} \leq 288.890439 = \overline{\mu}_4^{'}
\end{aligned} \qquad (30)$$

Using rules of interval arithmetics (8),(9), and (10) and intervals for raw moments (30), we have

$$\begin{aligned}
\mu_1 = [\underline{\mu}_1, \overline{\mu}_1] &= [\underline{\mu}_1^{'} - \overline{\mu}_1^{'}, \overline{\mu}_1^{'} - \underline{\mu}_1^{'}], \\
\mu_2 = [\underline{\mu}_2, \overline{\mu}_2] &= [\underline{\mu}_2^{'} - \overline{\mu}_1^{'2}, \overline{\mu}_2^{'} - \underline{\mu}_1^{'2}], \\
\mu_3 = [\underline{\mu}_3, \overline{\mu}_3] &= [\underline{\mu}_3^{'} - 3\overline{\mu}_2^{'}\overline{mu_1}^{'} + 2\underline{mu}_1^{'3}, \overline{\mu}_3^{'} - 3\underline{\mu}_2^{'}\underline{\mu}_1^{'} + 2\overline{\mu}_1^{'3}], \\
\mu_4 = [\underline{\mu}_4, \overline{\mu}_4] &= [\underline{\mu}_4^{'} - 4\overline{\mu}_3^{'}\overline{mu_1}^{'} + 6\underline{\mu}_2^{'}\underline{mu}_1^{'2} - 3\overline{mu_1}^{'4}, \\
&\quad \overline{\mu}_4^{'} - 4\underline{\mu}_3^{'}\underline{\mu}_1^{'} + 6\overline{\mu}_2^{'}\overline{mu_1}^{'2} - 3\underline{\mu}_1^{'4}],
\end{aligned} \qquad (31)$$

Using rules of interval arithmetics (8), (9), and (10) and equations (29) and (30), and intervals for central moments (31), we have lower and upper bounds for central moments of discrete binomial distribution as:

$$\begin{aligned}
\underline{\mu}_1 &= -8.16306899 & \leq \mu_1 \leq 8.16306899 = \overline{\mu}_1 \\
\underline{\mu}_2 &= -75.7280488 & \leq \mu_2 \leq 26.0176206 = \overline{\mu}_2 \\
\underline{\mu}_3 &= -692.277809 & \leq \mu_3 \leq 1450.66536 = \overline{\mu}_3 \\
\underline{\mu}_4 &= -21079.7244 & \leq \mu_4 \leq 12595.4731 = \overline{\mu}_4
\end{aligned} \qquad (32)$$

Using rules of interval arithmetics (8), (9), and (10) and equations (20) and (32), we have

$$\text{The coefficient of skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[\underline{\mu}_3, \overline{\mu}_3]^2}{[\underline{\mu}_2, \overline{\mu}_2]^3}$$
$$= \frac{[-692.277809, 1450.66536]^2}{[-75.7280488, 26.0176206]^3}$$
$$= \frac{[0, 1450.66536]^2]}{[-75.7280488^3, 26.0176206^3]} \quad (33)$$
$$= \frac{[0, 2104429.99]}{[-434280.472, 17611.7588]}$$
$$= [-\infty, 0/(-434280.472)] \cup [0/17611.7588, \infty]$$
$$= [-\infty, 0] \cup [0, \infty]$$
$$= [-\infty, \infty].$$

$$\text{The coefficient of kurtosis} = \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{[\underline{\mu}_4, \overline{\mu}_4]}{[\underline{\mu}_2, \overline{\mu}_2]^2}$$
$$= \frac{[-21079.7244, 12595.4731]}{[-75.7280488, 26.0176206]^2}$$
$$= \frac{[-21079.7244, 12595.4731]}{[0, -75.7280488^2]} \quad (34)$$
$$= \frac{[-21079.7244, 12595.4731]}{[0, 5734.73738]}$$
$$= [-\infty 12595.4731/5734.73738]$$
$$= [-\infty, 2.19634698].$$

7.2 Second Method

From Table 2, we have

$$\frac{\sum_{A\subseteq\Theta} Bel(A)}{\sum_{A\subseteq\Theta} p(A)} = (3/4)^{n-1} = (3/4)^{5-1} = (3/4)^4 = 0.31640625$$

$$\text{and } \frac{\sum_{A\subseteq\Theta} Pl(A)}{\sum_{A\subseteq\Theta} p(A)} = 2-(3/4)^{n-1} = 2-(3/4)^{5-1} = 2-(3/4)^4 = 1.68359375 \quad (35)$$

Using equations (25) and (35), we have,

$$(3/4)^{n-1}(\text{Stat.Quant.})_{pd} \le (\text{Stat.Quant.})_{pd} \le [2-(3/4)^{n-1}](\text{Stat.Quant.})_{pd}$$
$$\rightarrow (0-31640625)(\text{Stat.Quant.})_{pd} \le (\text{Stat.Quant.})_{pd} \le [1.68359375](\text{Stat.Quant.})_{pd} \quad (36)$$

Using the above equation (36), rules of operations on intervals (interval arithmetics) (8), (9), and (10), and raw moments of discrete binomial probability distribution under study, we have lower and upper bounds for raw moments as,

$$\begin{aligned}
\underline{\mu}_1' &= 0.84375 &\le \mu_1' \le 4.48958334 = \overline{\mu}_1' \\
\underline{\mu}_2' &= 0.53125 &\le \mu_2' \le 13.46875 = \overline{\mu}_2' \\
\underline{\mu}_3' &= 8.15625 &\le \mu_3' \le 43.3993056 = \overline{\mu}_3' \\
\underline{\mu}_4' &= 27.65625 &\le \mu_4' \le 147.158565 = \overline{\mu}_4'.
\end{aligned} \tag{37}$$

Using rules of interval arithmetics (8), (9), and (10) and lower and upper bounds for raw moments (37) and formulae for intervals for central moments (31), we have lower and upper bounds of central moments of discrete binomial distribution as

$$\begin{aligned}
\underline{\mu}_1 &= -3.64583334 &\le \mu_1 \le 3.64583334 = \overline{\mu}_1 \\
\underline{\mu}_2 &= -19.6251086 &\le \mu_2 \le 12.7568359 = \overline{\mu}_2 \\
\underline{\mu}_3 &= -172.049622 &\le \mu_3 \le 223.041882 = \overline{\mu}_3 \\
\underline{\mu}_4 &= -1968.29009 &\le \mu_4 \le 1746.99648 = \overline{\mu}_4.
\end{aligned} \tag{38}$$

Using rules of interval arithmetics (8), (9), and (10) and lower and upper bounds for central moments (38), we have

$$\begin{aligned}
\text{Coefficient of skewness} &= \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{[\underline{\mu}_3, \overline{\mu}_3]^2}{[\underline{\mu}_2, \overline{\mu}_2]^3} \\
&= \frac{[-0.172.049622, 223.041882]^2}{[-19.6251086, 12.7568359]^3} \\
&= \frac{[0, 223.041882^2]}{[-19.6251086^3, 12.7568359^3]} \\
&= \frac{[0, 49747.6811]}{[-7558.51025, 2076.00745]} \\
&= [-\infty, 0/(-7558.51025)] \cup [0/2076.00745, \infty] \\
&= [-\infty, 0] \cup [0, \infty] \\
&= [-\infty, \infty].
\end{aligned} \tag{39}$$

$$\begin{aligned}
\text{Coefficient of kurtosis} &= \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{[\underline{\mu}_4, \overline{\mu}_4]}{[\underline{\mu}_2, \overline{\mu}_2]^2} \\
&= \frac{[-1968.29009, 1746.99648]}{[-19.6251086, 12.7568359]^2} \\
&= \frac{[-1968.29009, 1746.99648]}{[0, -19.6251086^2]} \\
&= \frac{[-1968.29009, 1746.99648]}{[0, 385.144888]} \\
&= [-\infty, 1746.99648/385.144888] \\
&= [-\infty, 4.53594617].
\end{aligned} \tag{40}$$

8. APPLICATIONS

The intervals are the most appropriate approach or tool to deal with uncertainty. In the medical field, most of the medical parameters are represented in terms of intervals containing a single value of parameters of interest viz. Body-Mass Index, RBC count, WBC count, Sugar Level, Blood Pressure, Lymphocyte Counts, count, and so on. Using such medical parameters, medical analysis is performed consisting of statistical quantities viz. Range, Mean, Median, Mode, Variance, Standard Deviation, Skewness, Kurtosis, Correlation, Regression, Small Sample Tests, Large Sample Tests, and so on. Some processes of withdrawing conclusions with uncertainty are based on interval arithmetics. The approaches, mentioned in this paper, are useful to obtain lower and upper bounds of statistical quantities in terms of intervals. Therefore these approaches are helpful in the process of withdrawing conclusions with uncertainty. Also, using these approaches, we can obtain lower and upper bounds of several statistical quantities such as k^{th} ordered raw and central moments, measures of dispersion for the function of one variable.

9. CONCLUSION

Instead of having a single value representing statistical quantity, it is always better, to have an interval in which this single value is included. In this way, we include uncertainty regarding a single value representing statistical quantity. While obtaining lower and upper bounds of statistical quantities, we have taken care of getting more suitable bounds. This gives two approaches to finding lower and upper bounds of statistical quantities. The second method is more appealing than the first method and it gives closer lower and upper bounds of statistical quantities of discrete probability distribution under study. We hope that these approaches may be very useful in further research.

REFERENCES

[1] Bansi Lal; Sanjay Arora, " Mathematical statistics", *Satya Publications*, New Delhi, 1989.

[2] Deneoux Thierry, "Reasoning with Imprecise Belief Structures", *International Journal of Approximate Reasoning*, 20, Vol. 1,pp:79-111, 1999.

[3] Dempster A.P.," Upper and Lower Probabilities Induced by a Multivalued Mapping",*Annals of Mathematical Statistics*, Vol. 38, pp:325-339,1967.

[4] Dempster A.P.," Upper and Lower Probabilities Generated by a Random Closed Interval",*Annals of Mathematical Statistics*, Vol. 39, No.3, pp:957-966,1968.

[5] Ferson S.; Kreinovtch V.; Ginzburg L.; Myers D. S.; Sentz K., " Constructing Probability Boxes and Dempster-Shafer Structures", *Technical Report*, SAND2002-4015, Sandia National Laboratory, Albuquerque, Nm,2003.

[6] Hall H. S.; Knight S. R., â€œ Higher Algebraâ€ , *MacMilan & Co.*, New York,1891.

[7] Kandekar D. N., " A Note on Belief Functions Induced by Probability Distributions," *Proceedings of National Conference on Recent Advances in Mathematics (NCRAM-2017)*, pp. 68-82, ISBN 978-81-930245-4-5, 2017.

[8] Kandekar D. N., " New Belief Function Induced by Probability Function," *International Journal of Multidisciplinary Research and Advances in Engineering (IJPEM)*, Vol. 5, Issue 3, pp. 1-24, ISSN 0975-7074, 2017.

[9] Krieglar E.; Held H., "Utilizing Belief Functions for the Estimation of Uture Climate Change", *International Journal of Approximate Reasoning*, 39(2-3), pp:185-209, 2005.s

[10] Kyburg H. E., "Baysian and Non-Bayesian Updating", *Artificial Intelligence*, 31, pp271-294, 1987.

[11] Moore Ramon E., Kearfott R. Baker, and Cloud Michael J., " Introduction to Interval Analysis," *Society for Industrial and Applied Mathematics*, Philadephia, PA, USA, 2009.

[12] Shafer Glenn, " A Mathematical Theory of Evidence", *Princeton University Press*, NJ, 1976.

[13] Shafer Glenn, " Allocation of probability", *Annals of Probability*, V0l. 7, No. 5, pp:827-839, 1979.

[14] Yager Ronald R.," Dempster Shafer Belief Structures with Interval Valued Focal Weights", *International Journal of Intelligent Systems*, 16, Vol. 4, pp:497-512,2001.