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STUDY ON SOME INTEGRAL INEQUALITIES

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ABSTRACT:
Present paper studies some integral inequalities involving real valued functions and their derivatives. We try to generalise some inequalities which was introduced by Hardy, Littlewood and Polya.

KEYWORDS: *Integral inequality, Hilbert's invariant integral'.*

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§ 1: INTRODUCTION

Integral inequalities involving functions and their derivatives have been established by many authors in the literature during the past several years, see [3,4]. This paper is concerned with two theorems through which some new integral inequalities are introduced. One of them, in special case gives an existing inequality (1.1) which has been established in a book by Hardy, Littlewood and Polya [3, chapter VII, Theorem 254].

The existing inequality is:

If $\mu > 4$, $y(0) = 0$, $y(1) = 1$ and $y\,' \in L^2(0,1)$, then

(A) $$\int_0^1 \left\{\mu y'^2 - \frac{y^2}{x^2}\right\} dx > \frac{2}{1-2k} \qquad (1.1)$$

Where $k = \sqrt{(\frac{1}{4} - \frac{1}{\mu})}$

and finally we have $$\int_0^1 \left\{\mu y'^2 - \frac{y^2}{x^2}\right\} dx > 0$$

'Calculus Of Variations' plays a very important role to establish integral inequalities. From theory of 'Calculus Of Variations' we get 'Euler's equation' as necessary condition for existence of extremal Y [3,chapter VII ,7.1]which is

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \qquad (1.2)$$

Now Euler's equation (1.2) is applicable for functions of the form $J(y) = \int_{x_0}^{x_1} F(x, y, y')dx$.

One of the methods in 'Calculus of Variations' to prove integral inequalities is by using `Hilbert's invariant integral' which is

$$J^*(C) = \oint\{(F - pF_p)\,dx + F_p\,dy\} \qquad (1.3)$$

If the integral is taken along the extremal E , then using property of 'Hilbert's invariant integral' [3,chapter VII,7.5] we have

$$J(C) - J(E) \quad = \oint \in (x, y, p, y')dx \qquad (1.4)$$

Where

$$\in(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p) \qquad (1.5)$$

Where C is any curve in the region covered by the field and the integral is taken along C. Here F and Fp are the values of $F\,(x, y, y')$ and $F_{y'}\,(x, y, y')$ when y' is replaced by p and $\in$ is Weierstrass' exess function'. If $\in\, > 0$ whenever $y' \neq p$ then $J(E) < J(C)$ and E gives a true minimum of J.

Our first theorem is proved in two different methods . One method is done using the knowledge of transformation and 'Euler's equation' from Calculus of Variations. In another method we use `Hilbert's invariant integral' to prove the inequality. In Theorem 2 we also use Hilbert's invariant integral'which is an explicit application of 'Calculus Of Variations'.

Statement of Theorem 1 :

Theorem 1: If $y_0 = y(0) = k, k = \frac{b^m}{(a+b)^m}$, a $(\neq 0)$,b are real numbers , $m = \frac{1}{2} + \sqrt{\left(\frac{1}{4} - \frac{1}{a^2\mu}\right)}$, $\mu > \frac{4}{a^2}$, $y_1 = y(1) = 1, y' \in L^2(0,1)$,then

(B) $$\int_0^1 \left\{\mu y'^2 - \frac{y^2}{(ax+b)^2}\right\} dx = \frac{(a+b)^{2r} - b^{2r}}{a(a+b)^{1+2r}} \frac{2}{1-2r} + \mu \int_0^1 \left\{y' - \frac{a\left(\frac{1}{2}+r\right)y}{(ax+b)}\right\}^2 dx$$

Where $r = \sqrt{\left(\frac{1}{4} - \frac{1}{a^2\mu}\right)}$ and for a(>0) , b (≥ 0) we have the inequality

(B) $$\int_0^1 \left\{\mu y'^2 - \frac{y^2}{(ax+b)^2}\right\} dx > 0$$

Statement of Theorem 2 :

Theorem 2 : If $y_0 = y(0) = 0$, $y_1 = y(1) = 0$ and $y' \in L^2$ (0,1) , then

$$\int_0^1 \frac{y^2}{x(1-x)}\,dx < \frac{1}{2}\int_0^1 y'^2\,dx \ ,$$

unless $y = cx\,(1 - x)$

§2 : In this paper we give the first proof of Theorem 1 .

First proof of Theorem 1 :

Let $F(x, y, y') = \mu y'^2 - \frac{y^2}{(ax+b)^2}$

Then Euler's equation (1.2) becomes $\frac{-2y}{(ax+b)^2} = 2\mu y''$

$$i.e\ y'' + \frac{\lambda y}{(ax+b)^2} = 0 \qquad where\ \lambda = \frac{1}{\mu}, \text{ so } 0 < \lambda < \frac{4}{a^2} \qquad (2.1)$$

We take $(ax+b)$ =e^t, and $\theta \equiv \frac{d}{dt}$

Then equation (2.1) becomes $a^2\theta^2 y - a^2\,\theta + \lambda y = 0$ (2.2)

whose auxiliary equation is $a^2 l^2 - a^2 l + \lambda = 0$ (2.3)

Roots of the auxiliary equation (2.3) are

$$l = \frac{a^2+\sqrt{a^4-4a^2\lambda}}{2a^2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\lambda}{a^2}} \quad (2.4)$$

Case 1:

If $\frac{\lambda}{a^2} < \frac{1}{4}$, we have distinct roots of l. Then the general solution of (2.1) is

$$y = c_1\, e^{\left\{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\lambda}{a^2}}\right\}t} + c_2\, e^{\left\{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{\lambda}{a^2}}\right\}t}$$

$$= c_1\,(ax+b)^m + c_2(ax+b)^n \quad (2.5)$$

Where $m = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\lambda}{a^2}}$ (2.6)

and $n = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\lambda}{a^2}}$ (2.7)

Now $y' = \frac{dy}{dx} = c_1\, ma(ax+b)^{m-1} + c_2 na(ax+b)^{n-1}$ (2.8)

$y'^2 = c_1{}^2 m^2 a^2 (ax+b)^{2m-2} + c_2{}^2 n^2 a^2 (ax+b)^{2n-2} + 2c_1\,c_2 mna^2 (ax+b)^{-1}$ (2.9)

as $m+n=1$

$\int_0^1 y'^2 = \left[\frac{c_1{}^2 m^2 a(ax+b)^{2m-1}}{2m-1}\right]_1^0 + \left[\frac{c_2{}^2 n^2 a(ax+b)^{2n-1}}{2n-1}\right]_0^1 + \left[2c_1\,c_2 mna\,log(ax+b)\right]_0^1$ (2.10)

From (2.6) and (2.7) we have

$$2m-1 = 2\left(\sqrt{\frac{1}{4} - \frac{\lambda}{a^2}}\right) \qquad and \qquad 2\,n-1 = -2\left(\sqrt{\frac{1}{4} - \frac{\lambda}{a^2}}\right)$$

Since $\frac{\lambda}{a^2} < \frac{1}{4}$, 2m-1 is a positive quantity and 2n-1 is a negative quantity.

Hence from (2.10) the terms $\frac{c_2{}^2 n^2 a(ax+b)^{2n-1}}{2n-1}$ $and\ 2c_1\,c_2 mna\,log(ax+b)$ is undefined for x=0 and b=0.
So the integral $\int_0^1 y'^2$ in (2.10) only exists when $c_2 = 0$

Hence the general solution of (2.1) from (2.5) is $y = c_1\,(ax+b)^m$ (2.11)

Case2: If $\frac{\lambda}{a^2} = \frac{1}{4}$, we have equal roots for the equation (2.3) i.e $l = \frac{1}{2}$

Hence the general solution of (2.1) is $y(t) = (A+Bt)e^{\frac{t}{2}}$ (2.12)

Where A and B are arbitrary constants.

So $$y(x) = (A + Blog(ax + b))e^{\frac{1}{2}\log(ax+b)} \tag{2.13}$$

or $$y(x) = (ax + b)^{\frac{1}{2}}(A + Blog(ax + b)) \tag{2.14}$$

$$y' = \frac{1}{2}\, a(ax + b)^{-\frac{1}{2}}(A + Blog(ax + b)) + (ax + b)^{\frac{1}{2}} \frac{aB}{(ax+b)}$$

$$= a\left(\frac{A}{2} + B\right)(ax + b)^{-\frac{1}{2}} + \frac{aB}{2}(ax + b)^{-\frac{1}{2}}log(ax + b) \tag{2.15}$$

$$y'^2 = \frac{\left\{a\left(\frac{A}{2}+B\right)\right\}^2}{(ax+b)} + \frac{\left\{\frac{aB}{2}log(ax+b)\right\}^2}{(ax+b)} + a^2 B\left(\frac{A}{2} + B\right)(ax + b)^{-\frac{1}{2}}log(ax + b) \tag{2.16}$$

$$\int_0^1 y'^2 = \left[a\left\{\left(\frac{A}{2} + B\right)\right\}^2 \; log(ax + b)\right]_1^0 + \int_0^1 \frac{\left\{\frac{aB}{2}\,log(ax + b)\right\}^2}{(ax + b)}dx + a^2 B\left(\frac{A}{2} + B\right)\int_0^1 \left\{(ax + b)^{-\frac{1}{2}}log(ax + b)\right\}dx \tag{2.17}$$

`Now $log\,(ax + b)$ is not defined for $x = 0, b = 0$. So the integral $\int_0^1 y'^2$ in (2.17) does not exist. So in this case the solution 0f (2.1) does not exist.

So the only solution of (2.1) for which $y' \in L^2$ is $y = c_1\,(ax + b)^m$

Where $m = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\lambda}{a^2}}$

Using initial conditions If $y_0 = k$, $k = \frac{b^m}{(a+b)^m}$, $y_1 = 1$

$$c_1 b^m = \frac{b^m}{(a+b)^m} \tag{2.18}$$
$$c_1(a + b)^m = 1 \tag{2.19}$$

From (2.18) and (2.19) we get $c_1 = \frac{1}{(a+b)^m}$ (2.20)

So the extremal from (2.11) is $Y = \frac{1}{(a+b)^m}(ax + b)^m$ (2.21)

$$Y = \frac{1}{(a+b)^{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\lambda}{a^2}}}}(ax + b)^{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\lambda}{a^2}}} \text{ using (2.6)}$$

$$Y = \frac{1}{(a+b)^{\frac{1}{2}+r}}(ax + b)^{\frac{1}{2}+r} \tag{2.22}$$

Where $r = \sqrt{\frac{1}{4} - \frac{\lambda}{a^2}}$, $r^2 = \frac{1}{4} - \frac{\lambda}{a^2}$, $\mu = \frac{1}{a^2\left(\frac{1}{4} - r^2\right)}$, as $\lambda = \frac{1}{\mu}$ by (2.1) (2.23)

Then $$Y' = \frac{a\left(\frac{1}{2}+r\right)}{(a+b)^{\frac{1}{2}+r}}(ax + b)^{-\frac{1}{2}+r} \tag{2.24}$$

$$Y'^2 = \frac{\left\{a\left(\frac{1}{2}+r\right)\right\}^2}{(a+b)^{1+2r}}(ax+b)^{-1+2r} \tag{2.25}$$

Now $J(Y) = \int_0^1 \left\{\mu Y'^2 - \frac{Y^2}{(ax+b)^2}\right\} dx$

$$= \int_0^1 \left\{\mu \frac{\left\{a\left(\frac{1}{2}+r\right)\right\}^2}{(a+b)^{1+2r}}(ax+b)^{-1+2r} - \frac{\frac{1}{(a+b)^{1+2r}}(ax+b)^{1+2r}}{(ax+b)^2}\right\} dx \quad \text{using (2.22)}$$

$$= \frac{\left[\mu\left\{a\left(\frac{1}{2}+r\right)\right\}^2 - 1\right]}{(a+b)^{1+2r}} \int_0^1 (ax+b)^{-1+2r}\, dx$$

$$= \frac{\left[\mu\left\{a\left(\frac{1}{2}+r\right)\right\}^2 - 1\right]}{(a+b)^{1+2r}} \left[\frac{(ax+b)^{2r}}{2ra}\right]_0^1$$

$$= \frac{\left[\left(\frac{\frac{1}{2}+r}{\frac{1}{2}-r}\right)-1\right]}{(a+b)^{1+2r}} \left[\frac{(a+b)^{2r}-b^{2r}}{2ra}\right] \quad \text{using (2.1) } \mu = \frac{1}{a^2\left(\frac{1}{4}-r^2\right)}$$

$$= \frac{\left[\frac{\frac{1}{2}+r-\frac{1}{2}+r}{\frac{1}{2}-r}\right]}{(a+b)^{1+2r}} \left[\frac{(a+b)^{2r}-b^{2r}}{2ra}\right]$$

$$= \frac{\left[\frac{2r}{\frac{1-2r}{2}}\right]}{(a+b)^{1+2r}} \left[\frac{(a+b)^{2r}-b^{2r}}{2ra}\right]$$

$$= \frac{(a+b)^{2r}-b^{2r}}{a(a+b)^{1+2r}} \frac{2}{1-2r} \tag{2.26}$$

Now we take the transformation $y = Y + \eta$ (2.27)

$$J(y) = \int_0^1 \left\{\mu y'^2 - \frac{y^2}{(ax+b)^2}\right\} dx$$

$$= \int_0^1 \left\{\mu(Y'+\eta')^2 - \frac{(Y+\eta)^2}{(ax+b)^2}\right\} dx \quad \text{using (2.27)}$$

$$= \int_0^1 \left\{\mu Y'^2 - \frac{Y^2}{(ax+b)^2}\right\} dx + \int_0^1 \left\{\mu \eta'^2 - \frac{\eta^2}{(ax+b)^2}\right\} dx + K(Y,\eta)$$

$$= J(Y) + J(\eta) + K(Y,\eta) \tag{2.28}$$

Where $K(Y,\eta) = \int_0^1 \left\{2\mu Y'\eta' - \frac{2Y\eta}{(ax+b)^2}\right\} dx$

$$= 2\left\{[\mu Y'\eta]_0^1 - \mu \int_0^1 Y''\eta dx - \int_0^1 \frac{Y\eta}{(ax+b)^2} dx\right\} \tag{2.29}$$

Now Y' is a continuous function and $\eta(0) = \eta(1) = 0$ using (2.27) and given initial conditions. So $[\mu Y'\eta]_0^1 = 0$ then (2.29) becomes

$$K(Y,\eta) = -\mu\int_0^1 Y''\eta dx - \int_0^1 \frac{Y\eta}{(ax+b)^2}dx$$

$$= -\int_0^1 \left\{\mu Y'' + \frac{Y}{(ax+b)^2}\right\}\eta dx$$

Now Y is the solution of the equation (2.1) .So $\mu Y'' + \frac{Y}{(ax+b)^2} = 0$. Hence $K(Y,\eta) = 0$

so $J(y) = J(Y) + J(\eta)$ (2.30)

Again we take the transformation $\eta = Y\zeta$ (2.31)

$$J_\delta(\eta) = \int_\delta^1 \left\{\mu\eta'^2 - \frac{\eta^2}{(ax+b)^2}\right\}dx$$

$$= \int_\delta^1 \left\{\mu(Y'\zeta + Y\zeta')^2 - \frac{Y^2\zeta^2}{(ax+b)^2}\right\}dx$$

$= \int_\delta^1 \left\{\mu(Y'^2\zeta^2 + Y^2\zeta'^2 + 2YY'\zeta\zeta') - \frac{Y^2\zeta^2}{(ax+b)^2}\right\}dx$ (2.32)

Now

$\int_\delta^1 2YY'\zeta\zeta' dx = \int_\delta^1 2YY' d\zeta^2 = [YY'\zeta^2]_\delta^1 - \int_\delta^1 (Y'^2 + YY'')\zeta^2 dx$ (2.33)

Now $YY'\zeta^2 = \frac{Y\ Y'\eta^2}{Y^2} = \frac{Y'\eta^2}{Y}$ (2.34)

Now Y' is continuous and Y(1)=1 and η(1)=0,using initial conditions and the transformation (2.27) .So $YY'\zeta^2 = 0$ at x=1.

Now (2.32) becomes

$$J_\delta(\eta) = \int_\delta^1 \mu\left(Y'^2\zeta^2 + Y^2\zeta'^2\right)dx - \mu\left[YY'^{\zeta^2}\right]_\delta - \int_\delta^1 \mu\left(Y'^2 + YY''\right)\zeta^2 dx - \int_\delta^1 \frac{Y^2\zeta^2}{(ax+b)^{\ 2}}dx$$

using (2.32)

$= \int_\delta^1 \mu\left(Y^2\zeta'^2\right)dx - \mu\left[YY'^{\zeta^2}\right]_\delta - \int_\delta^1 \left(\mu Y'' + \frac{Y}{(ax+b)^{\ 2}}\right)Y\zeta^2 dx$

$= \int_\delta^1 \mu\left(Y^2\zeta'^2\right)dx - \mu[YY'\zeta^2]_\delta$ (2.35)

As Y is the solution of the equation (2.1) .So $\mu Y'' + \frac{Y}{(ax+b)^2} = 0$

$From\ (2.34)\quad YY'\zeta^2 = \frac{Y'\eta^2}{Y} = \frac{\frac{a\left(\frac{1}{2}+r\right)}{(a+b)^{\frac{1}{2}+r}}(ax+b)^{-\frac{1}{2}+r}\eta^2}{\frac{1}{(a+b)^{\frac{1}{2}+r}}(ax+b)^{\frac{1}{2}+r}}$ using (2.22) and (2.24)

$$= \frac{a\left(\frac{1}{2}+r\right)}{(ax+b)}\eta^2$$

For x=δ, $YY'\zeta^2 = \frac{a\left(\frac{1}{2}+r\right)}{(a\delta+b)}\eta^2(\delta)$ (2.36)

Case 1 : if b≠0, then $\lim_{\delta\to 0}[YY'\zeta^2]_\delta$ =0 from (2.36) as $\lim_{\delta\to 0}\eta(\delta)$=0

Case 2: if b=0, $\lim_{\delta\to 0}[YY'\zeta^2]_\delta = \lim_{\delta\to 0}\frac{a\left(\frac{1}{2}+r\right)}{(a\delta)}\eta^2(\delta) = \left(\frac{1}{2}+r\right)\lim_{\delta\to 0}\frac{\eta^2(\delta)}{\delta} = 0$

Since Y' and y' are in L^2, using transformation (2.27) η' is also in L^2 and so η=o($\sqrt{x}$) (theorem 222, Hardy, Littlewood, Polya)

So in all cases $\lim_{\delta\to 0}[YY'\zeta^2]_\delta = 0$ (2.37)

So $J(\eta) = \lim_{\delta\to 0}J_\delta(\eta) = \lim_{\delta\to 0}\int_\delta^1 \mu\left(Y^2\zeta'^2\right)dx - \mu\lim_{\delta\to 0}\left[YY'^{\zeta^2}\right]_\delta$

$= \int_0^1 \mu\left(Y^2\zeta'^2\right)dx$ using (2.37) (2.38)

Now from (2.31) we have $\eta = Y\zeta$

$$\eta' = Y'\zeta + Y\zeta'$$

So $Y\zeta' = \eta' - Y'\zeta = \eta' - \frac{a\left(\frac{1}{2}+r\right)}{(a+b)^{\frac{1}{2}+r}}(ax+b)^{-\frac{1}{2}+r}\frac{\eta}{\frac{(ax+b)^{\frac{1}{2}+r}}{(a+b)^{\frac{1}{2}+r}}}$ using (2.24) (2.31) and (2.22)

So $Y\zeta' = \eta' - a\left(\frac{1}{2}+r\right)\frac{\eta}{(ax+b)}$

$= (y' - Y') - a\left(\frac{1}{2}+r\right)\frac{(y-Y)}{(ax+b)}$ using (2.27)

$$= \left(y' - \frac{a\left(\frac{1}{2}+r\right)}{(a+b)^{\frac{1}{2}+r}}(ax+b)^{-\frac{1}{2}+r}\right) - \frac{a\left(\frac{1}{2}+r\right)\left(y - \frac{(ax+b)^{\frac{1}{2}+r}}{(a+b)^{\frac{1}{2}+r}}\right)}{(ax+b)}$$

using(2.24) and (2.22)

$= y' - \frac{a\left(\frac{1}{2}+r\right)}{(ax+b)}y$ (2.39)

Now from (2.30)we have $J(y) = J(Y) + J(\eta)$

$$= \frac{(a+b)^{2r}-b^{2r}}{a(a+b)^{1+2r}} \frac{2}{1-2r} + \mu \int_0^1 \left\{ y' - \frac{a\left(\frac{1}{2}+r\right)}{(ax+b)} y \right\}^2 dx \quad (2.40)$$

Using (2.38), (2.39) and (2.26)

Which is our required equality of Theorem 1.We also prove this equality by using `Hilbert's invariant integral'.

§3: Alternative proof of Theorem 1:

This proof is done by using `Hilbert's invariant integral'

Here $y = \frac{\alpha(ax+b)^{\frac{1}{2}+r}}{(a+b)^{\frac{1}{2}+r}}$ using (2.22) (3.1)

$p = \alpha \frac{a\left(\frac{1}{2}+r\right)}{(a+b)^{\frac{1}{2}+r}} (ax+b)^{-\frac{1}{2}+r}$ using (2.24)

$p = a\left(\frac{1}{2}+r\right)\frac{y}{ax+b}$ using (3.1) (3.2)

From §2 we have $F(x, y, y') = \mu y'^2 - \frac{y^2}{(ax+b)^2}$ (3.3)

$F_{y'} = 2\mu y'$ (3.4) ,

$F_p = 2\mu p = 2\mu a\left(\frac{1}{2}+r\right)\frac{y}{ax+b}$, using (3.2)

$= 2a \frac{1}{a^2\left(\frac{1}{4} - r^2\right)} \left(\frac{1}{2}+r\right)\frac{y}{ax+b}$ using (2.23)

$= \frac{2}{a\left(\frac{1}{2} - r\right)} \frac{y}{ax+b}$ (3.5)

Now $F - pF_p = \mu p^2 - \frac{y^2}{(ax+b)^2} - a\left(\frac{1}{2}+r\right)\frac{y}{(ax+b)}\frac{2}{a\left(\frac{1}{2} - r\right)} \frac{y}{(ax+b)}$

Using (3.2), (3.3), (3.5)

$= \frac{1}{a^2\left(\frac{1}{4} - r^2\right)}\left\{a\left(\frac{1}{2}+r\right)\frac{y}{ax+b}\right\}^2 - \frac{y^2}{(ax+b)^2} - \frac{\left(\frac{1}{2}+r\right)}{\left(\frac{1}{2} - r\right)} \frac{2y^2}{(ax+b)^2}$ using (2.23)

$= \frac{y^2}{(ax+b)^2}\left[\frac{\left(\frac{1}{2}+r\right)}{\left(\frac{1}{2} - r\right)} - 1 - 2\frac{\left(\frac{1}{2}+r\right)}{\left(\frac{1}{2} - r\right)}\right] = -\frac{y^2}{(ax+b)^2}\left[\frac{\left(\frac{1}{2}+r\right)}{\left(\frac{1}{2} - r\right)} + 1\right]$

$= -\frac{y^2}{\left(\frac{1}{2} - r\right)(ax+b)^2}$ (3.6)

Now from (1.5) $J^* = \int\left\{\left(F - pF_p\right)dx + F_p\, dy\right\}$

So

$J^* = \int\left\{-\frac{1}{\left(\frac{1}{2} - r\right)}\frac{y^2}{(ax+b)^2}\right\}dx + \left\{\frac{2}{a\left(\frac{1}{2} - r\right)} \frac{y}{ax+b}\right\}dy$ using (3.5) and (3.6)

$$= \frac{1}{a\left(\frac{1}{2} - r\right)} \int \left\{ \frac{-ay^2\,dx + 2y(ax+b)dy}{(ax+b)^2} \right\}$$

$$= \int \mathrm{d}\left(\frac{y^2}{a\left(\frac{1}{2} - r\right)(ax+b)} \right) \quad (3.7)$$

$= \int \mathrm{dW}$, a perfect differential, where $W = \left(\frac{y^2}{a\left(\frac{1}{2} - r\right)(ax+b)} \right)$

Now from (1.9) $\epsilon(x,y,p,y') = F(x,y,y') - F(x,y,p) - (y'-p)F_p(x,y,p)$

$$= \mu y'^2 - \frac{y^2}{(ax+b)^2} - \mu p^2 - \frac{y^2}{(ax+b)^2} - (y'-p)2\mu p$$

$$= \mu(y' - p)^2$$

$= \mu\left\{ y' - a\left(\frac{1}{2} + r\right)\frac{y}{ax+b} \right\}^2$ using (3.2) (3.8)

Now from (3.8) we have $J(C) - J(E) = \oint \epsilon\,(x,y,p,y')dx$

$= \int \mu\left\{ y' - a\left(\frac{1}{2} + r\right)\frac{y}{ax+b} \right\}^2$ using (3.8)

So $J(C) = \frac{(a+b)^{2r} - b^{2r}}{a(a+b)^{1+2r}}\frac{2}{1-2r} + \mu\int_0^1 \left\{ y' - \frac{a\left(\frac{1}{2}+r\right)}{(ax+b)}y \right\}^2 dx$ using (2.26) and(3.8) here $J(E) = J(Y)$,Y being the extremal and x=0 and x=1 being the end points of the extremal .

Hence $\int_0^1 \left\{ \mu y'^2 - \frac{y^2}{(ax+b)^2} \right\} dx = \frac{(a+b)^{2r} - b^{2r}}{a(a+b)^{1+2r}}\frac{2}{1-2r} + \int_0^1 \left\{ y' - \frac{a\left(\frac{1}{2}+r\right)}{(ax+b)}y \right\}^2 dx$ using (3.3) and (1.1)

§4: Now if $a(>0), b(\geq 0)$; $(a+b)^{2r} - b^{2r} > 0$,

$r = \sqrt{\frac{1}{4} - \frac{\lambda}{a^2}} > 0$ as $0 < \frac{\lambda}{a^2} < \frac{1}{4}$ using (2.1)

$$r = \sqrt{\frac{1}{4} - \frac{\lambda}{a^2}} < \frac{1}{2}$$

i.e $1 - 2r > 0$ .Hence $\frac{(a+b)^{2r} - b^{2r}}{a(a+b)^{1+2r}}\frac{2}{1-2r} > 0$ if $a(>0), b(\geq 0)$

so we have from the equality of Theorem 1

$$\int_0^1 \left\{ \mu y'^2 - \frac{y^2}{(ax+b)^2} \right\} dx > \mu \int_0^1 \left\{ y' - \frac{a\left(\frac{1}{2}+r\right)}{(ax+b)}y \right\}^2 dx$$

which gives the inequality of Theorem 1 i.e

$$\int_0^1 \left\{ \mu y'^2 - \frac{y^2}{(ax+b)^2} \right\} dx > 0$$

which completes the proof of Theorem 1.

§5: In this article we prove Theorem 2.

This inequality is also proved by using `Hilbert's invariant integral'.

Here we take $F(x, y, y') = \frac{1}{2}y'^2 - \frac{y^2}{x(1-x)}$ (5.1)

Then Euler's equation (1.2) becomes $\frac{-2y}{x(1-x)} = y''$

which gives $y'' + \frac{2y}{x(1-x)} = 0$

i.e $x(1-x)y'' + 2y = 0$ (5.2)

where the initial conditions are $y_0 = 0$, $y_1 = 0$
let $y = \alpha x(x-1)$ then $y' = \alpha(2x-1), y'' = 2\alpha$
then (5.2) has solutions $Y = \alpha x(x-1)$ (5.3)

Satisfying the initial conditions whatever be α.If E be a extremal passing through the end points(0,0)and (1,0) then E is of the from $Y = \alpha x(x-1)$.By varying α we can define a field round any particular extremal.

$$J(E) = \int_0^1 \left\{ \frac{1}{2} Y'^2 - \frac{Y^2}{x(1-x)} \right\} dx$$

$$J(E) = \int_0^1 \left\{ \frac{1}{2}\{\alpha(2x-1)\}^2 - \frac{\{\alpha x(x-1)\}^2}{x(1-x)} \right\} dx$$

$$J(E) = \alpha^2 \int_0^1 \left\{ \frac{1}{2}(4x^2 - 4x + 1) + x(x-1) \right\} dx$$

$= \alpha^2 \int_0^1 \left(3x^2 - 3x + \frac{1}{2}\right) dx = \alpha^2 \left(1 - \frac{3}{2} + \frac{1}{2}\right) = 0$

So $J(E) = 0$ for all α . (5.4)

From **§1** we have

$J(C) - J(E) = \oint \epsilon(x, y, p, y')dx$ using (1.4)

The integrals are taken along the curve C.

Where $\epsilon(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p)$ using (1.5)

Here $y = \alpha x(x-1)$ then $y' = p = \alpha(2x-1) = \frac{1-2x}{x(1-x)}y$ (5.5)

From (5.1) $F(x, y, y') = \frac{1}{2}y'^2 - \frac{y^2}{x(1-x)}$ $F_{y'} = y'$,so $F_p = p$ (5.6)

Now $F - pF_p = \frac{1}{2}p^2 - \frac{y^2}{x(1-x)} - p^2 = -\left(\frac{1}{2}p^2 + \frac{y^2}{x(1-x)}\right) = -\left\{\frac{1}{2}\left(\frac{1-2x}{x(1-x)}y\right)^2 + \frac{y^2}{x(1-x)}\right\}$ using (5.5)

$= -\left\{\frac{2y^2x(1-x)+(1-2x)^2y^2}{2x^2(1-x)^2}\right\}$ (5.7)

Now $(F - pF_p)\mathrm{dx} + F_p\, dy$

$= -\left\{\frac{2y^2x(1-x)+(1-2x)^2y^2}{2x^2(1-x)^2}\right\}dx + \frac{1-2x}{x(1-x)}y\, dy$ using (5.5)and (5.6)

$= -\frac{y^2}{2}\left\{\frac{2x(1-x)+(1-2x)^2}{2x^2(1-x)^2}\right\}dx + \frac{1-2x}{x(1-x)}y\, dy$

$= \frac{y^2}{2}\left\{\frac{x(1-x)d(1-2x)+(1-2x)d(x(1-x))}{x^2(1-x)^2}\right\} + \frac{1-2x}{2x(1-x)}2y\, dy$

$= y^2 d\left\{\frac{1-2x}{2x(1-x)}\right\} + \frac{1-2x}{2x(1-x)}\, d(y^2)$

$= d\left\{\frac{y^2(1-2x)}{2x(1-x)}\right\}$

$= dW$, a perfect differential, where $W = \left\{\frac{y^2(1-2x)}{2x(1-x)}\right\}$ (5.8)

Again from (1.9) we have

$\epsilon(x, y, p, y') = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p)$ (1.9)

$= \frac{1}{2}y'^2 - \frac{y^2}{x(1-x)} + \frac{1}{2}p^2 + \frac{y^2}{x(1-x)} - (y' - p)p$ using ((5.5) and (5.6)

$= \frac{1}{2}(y' - p)^2 > 0$, unless $y' = p$ (5.9)

So from (5.9) using (5.1), (1.8) and (5.4)

we get $\int_0^1 \left\{\frac{1}{2}y'^2 - \frac{y^2}{x(1-x)}\right\}dx > 0$ from which we get our required integral inequality

$\int_0^1 \frac{y^2}{x(1-x)}\, dx < \frac{1}{2}\int_0^1 y'^2\, dx$ (5.10)

$y' = p = \frac{dy}{dx} = \frac{1-2x}{x(1-x)}y$ using (5.5)

$\frac{dy}{y} = \frac{1-2x}{x(1-x)} = \frac{1-x-x}{x(1-x)}dx$

Solving we get

$Log\ y = log\ x\ + log\ (1 - x) + log\ c$ where c is arbitrary constants.

So Using (5.9) $y\ = c\ x\ (\ 1 - x\ )$ gives the equality of (5.10).

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