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ON SOME INTEGRAL INEQUALITIES INVOLVING THIRD ORDER DERIVATIVE

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ABSTRACT—
In the book "Inequalities" by Hardy, Little wood and Polyathe following inequality [chapter VII ,Theorem 259] has been proved.
If y and y'' are in $L^2(0, \infty)$then

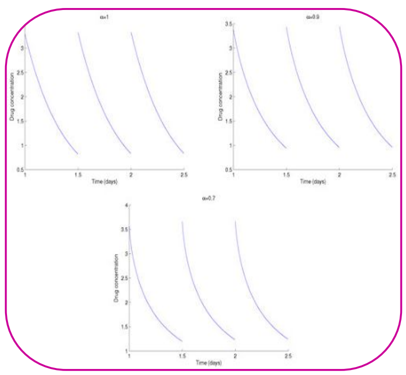
$$(\int_0^\infty y'^2 dx)^2 < 4\int_0^\infty y^2 dx \int_0^\infty y''^2 dx,$$

unless$y = AY(Bx)$, where A, B are constant and

$$Y = e^{-\frac{x}{2}} \sin(x sin\gamma - \gamma) \left(\gamma = \frac{\pi}{3}\right),$$

when there is equality.

In this paper we extend this inequality into an integral inequality involving third order derivative of y.



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§1INTRODUCTION:

In their book '*Inequalities*' Hardy, Little wood and Polya have proved the following inequality (1.1) [chapter VII, Theorem 259] involving second order derivative of y :

If y and $y"$ are in $L^2(0, \infty)$ then

$(\int_0^\infty y'^2 dx)^2 < 4\int_0^\infty y^2 dx \int_0^\infty y''^2 dx$,...(1.1)unless
$y = AY(Bx)$, ...(1.2)
where A and B are constants and
$Y = e^{-\frac{x}{2}} \sin(x sin\gamma - \gamma) \left(\gamma = \frac{\pi}{3}\right)$, ...(1.3)
when there is equality.

This inequality has been proved with the help of another inequality[2, chapter VII, Theorem 260] which is as follows :

If y and $y"$ are in $L^2(0, \infty)$ then

$\int_0^\infty (y^2 - y'^2 + y''^2)dx > 0$,……………………………………(1.4)

unless $y = AY$,…………………………………………………(1.5)

when there is equality.

Several proofs of (1.4) are given in the book [2, chapter VII, Theorem 260] to illustrate differences of method. The second proof is given by reducing (1.4) to dependence upon an identity.

In this paper we extend the inequality of (1.1) to an integral inequality where third order derivative of functions are involved. We also extend the inequality of (1.4) to an integral inequality involving third order derivative. With other conditions we take the condition$y'(0) = 0$ for these extensions. The extension of (1.1) is given in Theorem 1 which is statedbelow:

Theorem 1: If y, y''' and any one of y' or y'' are in $L^2(0, \infty)$ and $y'(0) = 0$ then $\left(\int_0^\infty y^2\,dx\right)\left(\int_0^\infty y'''^2\,dx\right) > (\int_0^\infty y'^2\,dx)(\int_0^\infty y''^2\,dx)$……..(1.6)

The extension of (1.4) is given in Theorem 2 which is stated below:

Theorem 2: If y, y''' and any one of y' or y'' are in $L^2(0, \infty)$ and $y'(0) = 0$ then $\int_0^\infty (y^2 - y'^2 - y''^2 + y'''^2)dx > 0$……………………………(1.6)

Theorem 2 is proved in §2 by reducing (1.7) to dependence upon an identity [2, chapter VII, Theorem 260].

In §3 first we prove Theorem 1 using Theorem 2. An alternative proof of Theorem 1 which is independent of Theorem 2 is givenlater.

Several examples of both Theorem 1 and Theorem 2 are given in §4.

In §5 we discuss the corrosponding theorems for complex valued functions in [0, ∞) and also for real valued functions in (−∞, ∞).

§2In this article we prove the Theorem 2.

The proof of Theorem 2:

Let $J_0 = \int_0^\infty y^2 dx$,$J_1 = \int_0^\infty y'^2 dx$,$J_2 = \int_0^\infty y''^2 dx$, and $J_3 = \int_0^\infty y'''^2 dx$where J_0, J_3 are finite and any one of J_1 or J_2 are finite.

If J_0, $J_2$and $J_3$are finite we can show $J_1$is finite in the same way that Hardy, Littlewood and Polya have done in the proof of (1.4) [2, chapter VII, Theorem260]

Now we assume J_0, J_1, $J_3$are finite and we show $J_2$is finite.

We have

$\int_0^X y''^2 dx = y'y''|_0^X - \int_0^X y'y'''dx$.....................................(2.1)

Since y' and y''' are in $L^2(0, \infty)$, $\int_0^X y'y'''dx$ tends to a finite limit as X tends to ∞. If J_2 $i.e.$ $\int_0^\infty y''^2 dx$ were infinite.$y'y''(X)$would tend to ∞ as X tends to ∞. But,

$$y'^2 = 2\int y'y''dx,$$

which gives if $y'y''$ is infinite then y'^2 is infinite which contradicts the convergence of J_1.
Hence$J_2$is finite.
Therefore, under the given conditions J_0, J_1, J_2, J_3 are finite.
Now

$$\int_0^X [(y^2 - y'^2 - y''^2 + y'''^2) - (y + y' + y'' + y''')^2]dx$$

$$= -2\int_0^X (y'^2 + y''^2 + yy' + yy'' + yy''' + y'y'' + y'y''' + y''y''')dx$$

$$= -2\int_0^X [(y + y' + y'')(y' + y'' + y''') - y'y'']dx$$

$$= -\int_0^X d(y + y' + y'')^2 + \int_0^X d(y')^2$$

$$= -(y + y' + y'')^2|_0^X + y'^2|_0^X$$

$$= -(y + y' + y'')^2(X) + (y + y' + y'')^2(0) + y'^2(X) - y'^2(0) \qquad (2.2)$$

Now J_0, J_1, J_2, J_3 being finite, $(y + y' + y'')^2(X)$ and $y'^2(X)$ tend to zero as X tends to infinity.

When $X \to \infty$, (2.2) becomes

$$\int_0^\infty [(y^2 - y'^2 - y''^2 + y'''^2) - (y + y' + y'' + y''')^2]dx$$

$= (y + y' + y'')^2(0) - y'^2(0)$..(2.3)

We apply the condition $y'(0) = 0$ in (2.3) and get

$$\int_0^\infty (y^2 - y'^2 - y''^2 + y'''^2)dx$$

$= [y(0) + y''(0)]^2 + \int_0^\infty (y + y' + y'' + y''')^2 dx$..................(2.4)

Since all the terms in the R.H.S of (2.4) arepositive,

$$\int_0^\infty (y^2 - y'^2 - y''^2 + y'''^2)dx > 0$$

which is the required integral inequality(1.7).
Equality occurs in (1.7) when
$y(0) + y''(0) = 0$..(2.5)
And
$y + y' + y'' + y''' = 0$..(2.6)

Solving (2.6) we get three linearly independent solutions of the form $y_1 = e^{-x}, y_2 = cosx$ and $y_3 = sinx$ from which only e^{-x} is in $L^2(0,\infty)$, but for $y = e^{-x}$ the equation (2.5) and $y'(0) = 0$ are not satisfied. So under the given conditions strict inequality follows in (1.7).

§3 In this article we prove Theorem 1 in different ways.

The proof of Theorem 1 :

In order to deduce Theorem 1 from Theorem 2 ,we apply Theorem 2 to $z(x)$instead of $y(x)$, where $z(x) = y\left(\frac{x}{\rho}\right)$, ρ is a positive quantity.

Then (1.7) becomes

$$\int_0^\infty (z^2 - z'^2 - z''^2 + z'''^2)dx > 0 \dots\dots\dots\dots\dots\dots\dots (3.1)$$

For $z(x) = y\left(\frac{x}{\rho}\right)$

$$z'(x) = \left(\frac{1}{\rho}\right)y'\left(\frac{x}{\rho}\right), \quad z''(x) = \left(\frac{1}{\rho^2}\right)y''\left(\frac{x}{\rho}\right), \quad z'''(x) = \left(\frac{1}{\rho^3}\right)y'''\left(\frac{x}{\rho}\right)$$

Then (3.1) becomes

$$\int_0^\infty \left\{\rho^6 y^2\left(\frac{x}{\rho}\right) - \rho^4 y'^2\left(\frac{x}{\rho}\right) - \rho^2 y''^2\left(\frac{x}{\rho}\right) + y'''^2\left(\frac{x}{\rho}\right)\right\}dx > 0 \dots\dots\dots\dots (3.2)$$

Let $\frac{x}{\rho} = t$, then (3.2) becomes

$$\int_0^\infty \{\rho^6 y^2(t) - \rho^4 y'^2(t) - \rho^2 y''^2(t) + y'''^2(t)\}\rho dt > 0 \dots\dots\dots\dots (3.3)$$

Now ρ being positive,

$$\rho^6 \int_0^\infty y^2(t)dt - \rho^4 \int_0^\infty y'^2(t)dt - \rho^2 \int_0^\infty y''^2(t)dt + \int_0^\infty y'''^2(t)dt > 0 \dots\dots\dots\dots\dots\dots\dots (3.4)$$

which gives

$$\rho^6 J_0 - \rho^4 J_1 - \rho^2 J_2 + J_3 > 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3.5)$$

Since (3.5) is true for any positive quantity ρ, we now take $\rho = \sqrt{\frac{J_3}{J_2}}$

$i.e. \rho^2 = \frac{J_3}{J_2}$and we get from (3.5)

$$\frac{J_3^3}{J_2^3}J_0 - \frac{J_3^2}{J_2^2}J_1 - \frac{J_3}{J_2}J_2 + J_3 > 0$$

$\Rightarrow \frac{J_3^3}{J_2^3} J_0 - \frac{J_3^2}{J_2^2} J_1 > 0$

$\Rightarrow J_3 J_0 - J_2 J_1 > 0$

As$\frac{J_3^2}{J_2^2} > 0$.

So we have
$J_0 J_3 > J_1 J_2 \Longrightarrow \int_0^\infty y^2 dx \int_0^\infty y'''^2 dx > \int_0^\infty y'^2 dx \int_0^\infty y''^2 dx$which is the required integral inequality (1.6).

Alternative proof ofTheorem 1 :

Let $J = \int_0^\infty yy' dx$ and $H = \int_0^\infty y' y''' dx$.

Now we have

$\int_0^X yy'' dx = yy'|_0^X - \int_0^X y'^2 dx$ ....(3.6)

As $X \to \infty$ we apply the convergence of J_0, J_1 and further apply the given condition $y'(0) = 0$ and get from (3.6)

$J = -J_1$..(3.7)

Proceeding in a similar way for $y'$and y'''we get

$H = -J_2$..(3.8)

Applying Cauchy-Schwarz'sine quality we get

$J^2 \le J_0 J_2$..(3.9)

Equality occurs when $y = De^{-x}$,......................................(3.10)

where D is any arbitrary constant.

Applying Cauchy-Schwarz's inequality we get

$$H^2 \le J_1 J_3 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (3.11)$$

Equality occurs when $y = De^{-x}$....................................(3.12)

where D is an arbitrary constant.

Multiplying respective sides of (3.9) and (3.11) we get

$J^2 H^2 \le J_0 J_2 J_1 J_3$...(3.13) equality occurs when $y = De^{-x}$.......................................(3.14)

Using (3.7), (3.8) and (3.13) we get

$J_1^2 J_2^2 < J_0 J_2 J_1 J_3$...(3.15)

We note that for $y = e^{-x}$ the equation (3.7) and (3.8) are not true as the given condition $y'(0) = 0$ is not satisfied. Hence under the given conditions strict inequality follows in (3.15).

So we have from (3.15)

$$J_3 J_0 > J_1 J_2 \Rightarrow \int_0^\infty y^2 dx \int_0^\infty y'''^2 dx > \int_0^\infty y'^2 dx \int_0^\infty y''^2 dx$$

which is the required integral inequality (1.6).

§4 In the article we verify our inequalities (1.6) and (1.7) by taking different examples. We will explain Example 1 in detail and mention other some examples at the end.

Example 1: Let $y = e^{-\frac{x}{\sqrt{2}}} sin\left(\frac{\pi}{4} + \frac{x}{\sqrt{2}}\right) \qquad x \in [0, \infty)$...........................(4.1)

Differentiating (4.1) twice and thrice we get

$y'' + \sqrt{2}y' + y = 0$..(4.2)

$y''' + \sqrt{2}y'' + y' = 0$..(4.3)

respectively.

After conclusion it can be shown that $y,\ y',\ y'',\ y'''$ all are in $L^2(0,\ \infty)$and $y'(0) = 0$.
Now we consider the expression

$$\int_0^X (y^2 - y'^2 - y''^2 + y'''^2) dx$$

Using (4.3) the above expression

$$= \int_0^X \{y^2 - y'^2 - y''^2 + (\sqrt{2}y'' + y')^2\} dx$$

$$= \int_0^X (y^2 + y''^2) dx + \sqrt{2} \int_0^X d(y'^2)$$

$= \int_0^X (y^2 + y''^2) dx + \sqrt{2}[y'^2(X) - y'^2(0)]$........................(4.4)

Now since y' is in $L^2(0, \infty)$ and $y'(0) = 0$ then as X tends to ∞ (4.4) becomes

$\int_0^X (y^2 - y'^2 - y''^2 + y'''^2)dx = \int_0^X (y^2 + y''^2)dx > 0$……………..(4.5)

As y is not identically zero.
Hence (1.7) is verified.
Now we consider the expression

$$\int_0^X y^2 dx \int_0^X y'''^2 dx$$

Using (4.2) and (4.3) the above expression

$$= \int_0^X (y'' + \sqrt{2}y')^2 dx \int_0^X (\sqrt{2}y'' + y')^2 dx$$

$= \int_0^X (y''^2 + 2\sqrt{2}y''y' + 2y'^2)dx \int_0^X (2y''^2 + 2\sqrt{2}y''y' + y'^2)dx$……………(4.6)

Now

$2\int_0^X y'y''dx = y'^2(X) - y'^2(0)$…………………………………………………(4.7)

Using the given conditions as X tends to ∞ the R.H.S. of (4.7) tends to zero.
Now when X tends to ∞ (4.6) becomes

$$\int_0^\infty y^2 dx \int_0^\infty y'''^2 dx$$

$$= \int_0^\infty (y''^2 + 2y'^2)dx \int_0^\infty (2y''^2 + y'^2)dx$$

$= 2\left(\int_0^\infty y''^2 dx\right)^2 + \int_0^\infty y''^2 dx \int_0^\infty y'^2 dx + 4\int_0^\infty y'^2 dx \int_0^\infty y''^2 dx + 2\left(\int_0^\infty y'^2 dx\right)^2$…(4.8)

Now (4.8) gives

$$\left(\int_0^\infty y^2 dx\right)\left(\int_0^\infty y'''^2 dx\right) > \left(\int_0^\infty y'^2 dx\right)\left(\int_0^\infty y''^2 dx\right)$$

as$y$ is not identically zero.
Hence (1.6) is verified.
Other some examples are:

2. $y = e^{-\frac{x}{\sqrt{2}}} cos\left(-\frac{\pi}{4} + \frac{x}{\sqrt{2}}\right) \qquad x \in [0, \infty)$
3. $y = e^{-\frac{x}{2}} sin\left(\frac{\pi}{6} + \frac{x}{2\sqrt{3}}\right) \qquad x \in [0, \infty)$

4. $y = e^{-\frac{x}{\sqrt{2}}} cos\left(\frac{\pi}{4} - \frac{x}{\sqrt{2}}\right) \qquad x \in [0, \infty)$

5. $y = e^{-\frac{x}{\sqrt{3}}} cos\left(\frac{\pi}{3} + x\right) \qquad x \in [0, \infty)$

Note: We note that for $y = e^{-x}$, $y, y', y'', y''' \in L^2(0, \infty)$ and equality occurs in both (1.6) and (1.7) but the given condition $y'(0) = 0$ does not hold.

§5.Remark 1: We observe that if f is a complex valued function$i.e. f = f_1 + if_2$, where f_1 and $f_2$are the real and imaginary part of $f$respectively, then $|f|^2 = {f_1}^2 + {f_2}^2$. f', f''andf''' can be expressed in the similar way.

We now apply Theorem 2 separately on both real and imaginary parts of f and its derivatives and then adding the two results we get the following theorem corresponding to Theorem 2for a complex valued function f.

Theorem 4: For a complex valued function f, if$|f|$, $|f'''|$ and any one of $|f'|$ or $|f''|$ are in $L^2(0, \infty)$ and $f'(0) = 0$ then

$J(|f|) = \int_0^\infty \{|f|^2 - |f'|^2 - |f''|^2 + |f'''|^2\} dx > 0$……………….(5.1)

Similarly we have Theorem 3 for complex valued function f, which is the theorem corresponding to Theorem 1.

Theorem 3: For a complex valued function f, if $|f|$, $|f'''|$ and any one of $|f'|$ or $|f''|$ are in $L^2(0, \infty)$ and $f'(0) = 0$ then

$\left(\int_0^\infty |f|^2 dx\right)\left(\int_0^\infty |f'''|^2 dx\right) > \left(\int_0^\infty |f'|^2 dx\right)\left(\int_0^\infty |f''|^2 dx\right)$…………………(5.2)

We can prove Theorem 3 from Theorem 4 in the same way as Theorem 1 is proved from Theorem 2.

Remark 2: It is interesting to observe that when the inequality is considered over the interval $(-\infty, \infty)$ we do not require the condition $y'(0) = 0$ for corresponding theorems of Theorem 1 and Theorem 2. The corresponding Theorem 1 and Theorem 2 are Theorem 5 and theorem 6 respectively which are as follows:

Theorem 5: If y, y''' and any one of y' or y'' are in $L^2(-\infty, \infty)$ then
$\left(\int_{-\infty}^\infty y^2 dx\right)\left(\int_{-\infty}^\infty y'''^2 dx\right) > \left(\int_{-\infty}^\infty y'^2 dx\right)\left(\int_{-\infty}^\infty y''^2 dx\right)$………..(5.3)
unless $y = DY$……………………………………………………………(5.4)
where D is an arbitrary constant and
$Y = e^{-x}$,…………………………………………….(5.5)
when there is equality.
Theorem 6: If y, y''' and any one of y' or y'' are in $L^2(-\infty, \infty)$ then
$\int_{-\infty}^\infty (y^2 - y'^2 - y''^2 + y'''^2) dx > 0$………………………(5.6)
unless $y = DY$……………………………………………………………(5.7)
where D is an arbitrary constant.

Proof of Theorem 6: Proceeding in the same way as the proof of Theorem 2 on $[-X,X]$we have

$$\int_{-X}^{X}[(y^2-y'^2-y''^2+y'''^2)-(y+y'+y''+y''')^2]dx$$

$= -(y+y'+y'')^2|_{-X}^{X}+y'^2|_{-X}^{X}$……………………………………(5.8)

Now from the given conditions, $(y+y'+y'')^2(X),\ y'^2(X)$ and $(y+y'+y'')^2(-X),\ y'^2(-X)$ tends to zero as X tends to infinity.
Hence when $X\to\infty$ we have

$$\int_{-\infty}^{\infty}(y^2-y'^2-y''^2+y'''^2)dx$$

$= \int_{-\infty}^{\infty}(y+y'+y''+y''')^2dx$……………………………….(5.9)

So we have

$$\int_{-\infty}^{\infty}(y^2-y'^2-y''^2+y'''^2)dx > 0$$

which is the required inequality (5.6).
Equality occurs when$y+y'+y''+y'''=0$, which is the equation (2.6)

Following the solution of (2.6) in §2 we get the only solution in $L^2(-\infty,\infty)$ as $y=e^{-x}$ which gives the equality case here.

Theorem 5 can also be proved from Theorem 6 in the similar way as Theorem 1 is proved from Theorem 2.

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