REVIEW OF RESEARCH
IMPACT FACTOR : 5.7631(UIF) UGC APPROVED JOURNAL NO. 48514 ISSN: 2249-894X
VOLUME - 8 | ISSUE - 4 | JANUARY - 2019

SIMPLICITY OF SIMPLE GROUPS WITH SYLOW'S THEOREMS

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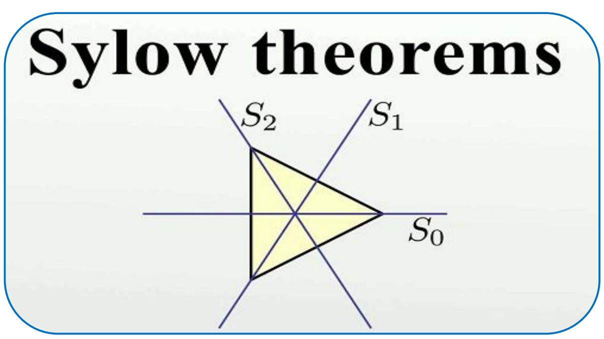
ABSTRACT:

One of the important results in the theory of finite groups is Lagrange's theorem, which states that the order of any subgroup of a group must divide the order of the group. i.e., If H is a subgroup of a finite group of order n then according the Lagrange's theorem O(H) divides O(G). But the converse to this theorem is may or may not be true that is for any number d dividing the order of a group then there may not be exists a subgroup of G of order d. The simplest example of this is the group A_4 of even permutations on the set {1,2,3,4} has order 12, yet there does not exist a subgroup of order 6. The Norwegian mathematician Peter Ludwig Sylow discovered that the converse of Lagrange's theorem is true when d is a prime power. If p is a prime number and P^k |O(G).Then G must contains a subgroup of order P^k. Sylow also discovered important relationships among the subgroups whose order is the largest power of p dividing O(G), such as the fact that all subgroups of that order are conjugate to each other. The aim of the paper is to present the Sylow theorems in easy manner and their applications.

KEYWORDS : *Lagrange's theorem , Norwegian mathematician.*

INTRODUCTION

1.1 Even Perfect Number: A perfect even number is an even number which is perfect .In number theory, a perfect number is a positive integer that is equal to the sum of its positive divisors, excluding the number itself. For example, 6 has divisors 1, 2 and 3 (excluding itself), and 1 + 2 + 3 = 6, so 6 is a even perfect number. Even perfect number is always of type $N = 2^{p-1}\left(2^p - 1\right)$. *For Example* $28 = 2^{3-1}\left(2^3 - 1\right) = 2^2.7$, $496 = 2^{5-1}\left(2^5 - 1\right) = 2^4.31$ and $8128 = 2^{7-1}\left(2^7 - 1\right) = 2^6.127$ all are even perfect number.



1.2 Group: A non empty set G equipped with an operation '*' is said to be group if it satisfies the following postulates.

1. **Closure Property:** *i.e.* $\forall \; a,b \in G \Rightarrow a*\bar{b} \in G.$
2. **Associativity :** *i.e.* $a*(b*c)=(a*b)*c \quad \forall a,b,c \in G.$
3. **Existence of identity element:** If $a \in G$ the there exists an element $e \in G$ such that

$$e*a=a=a*e \quad \forall a \in G.$$

4. **Existence of inverse element:** If $a \in G$ the there exists an element $a^{-1} \in G$ such that

$$a^{-1}*a=e=a*a^{-1} \quad \forall a \in G.$$

1.3 Finite and Infinte Group: A group G is said to be finite group if the number of distinct elements in group is finite otherwise G is an infinite group.

1.4 Order of Group: The number of elements in a finite group is called the order of the group and is denoted by $O(G)$. An infinite group is said to be of infinite order.

1.5 Order of an element in a Group: Let G be a group and $a \in G$. If there exists a least positive inter n such that $a^n = e$

Then n is called the order of a and written as $O(a)=n$. If no such positive integer exists, then a is said to be of infinite order.

1.6 Subgroup: A non empty subset H of a group G is called a subgroup of G if H itself is a group w.r.t. the same composition as defined in G.

1.7 Cosets: Let H be a subgroup of a group G and $a \in G$. The set

$aH = \{ ah \; : \; h \in H \}$ is called a left coset of H in G.
$Ha = \{ ha \; : \; h \in H \}$ is called a right coset of H in G.

1.8 Normal Subgroup: A subgroup H of a group G is said to be normal subgroup of G if

$$xH = Hx \qquad \forall \; x \in G.$$

1.9 Cauchy's Theorem: If G is a finite group and p is a prime number. If $P|O(G)$ then there exists an element a in G such that $O(a)=p$.

Remarks-

1. If G is a finite group and $P|O(G)$, p is a prime number, then G has a subgroup of order p.
2. If G is a finite group and $m|O(G)$, m is a positive integer, then G has a subgroup of order m.

1.10 Lagrange's Theorem: If G is a finite group and H is a subgroup of G then *O(H)|O(G).*

1.11 Normalize of a subgroup a group: If H is subgroup of a group G then normalize *N(H)* of H in G is the set of all those elements *x* in G that commute with *H. i.e.*

$$N(H)=\{x\in G\ :\ xH=Hx\}$$

1.12 Conjugate Subgroup: If H and K are any two subgroups of a group G then H is said to be conjugate to K if there exists $g\in G\quad such\ that\ gHg^{-1}=K$.

1.13 Simple Group: A group having no proper normal subgroups is called a simple group.

1.14 p- Groups: A group G is said to be p- group (p being a prime number) if the order of every elements of G is some power of p. In other words we can say that a finite group G is a p- group iff $O(G)=p^n\quad ,p\ being\ a\ prime\ number.$

Example: The Quaterian group G of order 8 is a p- group.

1.15 Theorem: The order of a subgroup of a finite group divides the order of the group.

Proof: Let G be a finite group of order n and H be a subgroup of order m $i.e., O(G)=n\ and\ O(H)=m$ then we shall prove that m|n . For,

Since O(H)= m. Therefore there exist m members in H. Let $h_1,h_2,...,h_m$ are the m members of H. Let a □ G then $Ha=\{h_1a,\ h_2a,...,h_ma\}$ is a right coset of H having m distinct members, since $h_ia=h_ja\Rightarrow h_i=h_j$.

Therefore each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint that is they have no element in common. Since G is a finite group, therefore the number of distinct right cosets of H in G will be finite, say, equal to k. The union of these k distinct right cosets of H in G is equal to G. Thus if $Ha_1,Ha_2,\ ...,\ Ha_k$ are the k distinct right cosets of H in G then

$$G=Ha_1\cup Ha_2\cup\ ...\cup Ha_k$$
$$\Rightarrow\ O(G)=O(Ha_1)+O(Ha_2)+...+O(Ha_k)$$
$$\Rightarrow n=m+m+...+m\,(k\ times)$$
$$\Rightarrow n=mk$$
$$\Rightarrow m|n$$

SYLOW'S THEOREM

2.1 SYLOW'S FIRST THEOREM: If G be a group and p be a prime number such that $P^m \mid O(G)$ Then G has a subgroup of order P^m.

Proof: Let G be a group of order n and p be a prime number $P^m \mid O(G)$ then we shall prove that G has a subgroup of order P^m. For, We shall prove this theorem by induction on n.

If $n=1$ then $G=\{e\}$ has a subgroup of order $P^0=1$ *i.e.,* $\{e\}$ itself. Thus the result is true for $n=1$.

Now, let us suppose that the result is true for all groups of order less than n. It means if L is any group of order less than n and if $P^k \mid O(L)$ then L has a subgroup of order P^k.

Since $P^m \mid O(G)$ then there arise two cases

Case 1: Let P^m divide the order of a proper subgroup H of G i.e., $P^m \mid O(H)$,where $H< G$. Therefore by induction hypothesis H has a subgroup of order P^m. But $H< G$, therefore G has a subgroup of order P^m.

Case 2: Let $P^m \nmid O(H)$ for all proper subgroup of H of G. The class equation of G is

$$O(G)=O(Z)+\sum_{a\notin Z}\frac{O(G)}{O[N(a)]}$$

Or,

$$O(G)=O(Z)+\sum_{N(a)\neq G}\frac{O(G)}{O[N(a)]} \qquad ...(1)$$

where the summation runs over one lelment a in each conjugate class containing more than two element.

we have

$$p^m \mid O(G)$$

$$\Rightarrow p^m \left| \frac{O(G)}{O[N(a)]} \quad O[N(a)]\right.$$

$$\Rightarrow p^m \left| \frac{O(G)}{O[N(a)]}\right.$$

< Since $P^m \nmid O[N(a)]$ as $N(a)< G$ and $N(a) \neq G$ >

$$\Rightarrow p^m \left| \sum_{N(a)\neq G} \frac{O(G)}{O[N(a)]} \right.$$

$$\Rightarrow p \left| \sum_{N(a)\neq G} \frac{O(G)}{O[N(a)]} \right. \qquad \left\{ Since \quad \sum_{N(a)\neq G} \frac{O(G)}{O[N(a)]} = p^l \quad , where\ 0 < l \leq m > \right\}$$

$$\Rightarrow p \left| \left(O(G) - \sum_{N(a)\neq G} \frac{O(G)}{O[N(a)]} \right) \right. \qquad \{Since \quad p \mid O(G)\}$$

$$\Rightarrow p \mid O(Z)$$

Since $P \mid O(Z)$. Therefore by Cauchy's theorem for abelian groups there exists some integer $a \neq e \in Z$ such that $a^p = e$. Consequently $K = <a> = \{a, a^2, a^3, ..., a^p = e\}$ is a subgroup of Z and $O(Z)=p$.

Now, $K < Z \Rightarrow K$ *is a normal subgroup of* G *and so* $O\left(\frac{G}{K}\right) = \frac{O(G)}{O(K)} < O(G)$

Further $p^{m-1} \left| O\left(\frac{G}{K}\right) \right.$, *Since* $O(K) = p$ *and* $p^m \mid O(G)$.

Since $p^{m-1} \left| O\left(\frac{G}{K}\right) \right.$. *Therfore by induction hypothesis,* $\frac{G}{K}$ *has a subgroup, say,* $\frac{H}{K}$, *of order* p^{m-1}.

i.e., $O\left(\frac{H}{K}\right) = p^{m-1}$

Thus $O(H) = O\left(\frac{H}{K}\right) O(K)$

$$= p^{m-1}.p$$

$$= p^m$$

Hence $O(H) = p^m$ *and* $H < G$

This completes the induction and theorem is proved.

2.2 Sylow p- subgroup: Let H is any subgroup of a finite group G such that $O(H)= p^m$, p being a prime number then H is called Sylow p-subgroup or p-sylow subgroup or p-SSG if $p^m \mid O(G)$ and $p^{m+1} \nmid O(G)$.

2.3 SYLOW'S SECOND THEOREM: Any two sylow p-subgroups of a finite group G are conjugate in G.

2.4 SYLOW'S THIRD THEOREM: The number of p-sylow subgroups of a finite group G is always congruent to 1 modulo p and divides order of group. i.e., If there are n_p , *P-SSG* then

$$n_p \equiv 1 \pmod p \quad and \quad n_p \mid O(G)$$

$$\Rightarrow p \mid (n_p - 1) \quad and \quad n_p \mid O(G)$$

$$\Rightarrow n_p - 1 = pt \quad and \quad n_p \mid O(G), \qquad where\ t = 0,\ 1,\ 2, \ldots$$

$$\Rightarrow n_p = 1 + pt \quad and \quad n_p \mid O(G)$$

Thus total number of p*-* $SSG = 1 + pt$ *and* $1 + pt \mid O(G)$.

2.5 EXAMPLE: If G be a finite group of order 144

i.e. $O(G) = 144 = 2^4.3^2$

Since 2^4 *|144 and* $2^{4+1} \nmid 144 \Rightarrow G$ *has a* 2*-SSG of order* $2^4 = 16$

and *number of* $2\text{-}SSG = 1 + 2t,\ t = 0,1,2,\ldots$ *and* $1 + 2t \mid 2^4.3^2$

$\Rightarrow$ *number of* $2\text{-}SSG = 1 + 2t,\ t = 0,1,2,\ldots$ *and* $1 + 2t \mid 3^2$, *Since* $\gcd(1+2t,\ 2^4) = 1$

$\Rightarrow$ *number of* $2\text{-}SSG = 1 + 2t,\ t = 0,1,2,\ldots$ *and* $1 + 2t \mid 9$

$\Rightarrow$ *number of* $2\text{-}SSG = 1,3,9,\ t = 0,1,4.$

Again,

Since 3^2 *|144 and* $3^{2+1} \nmid 144 \Rightarrow G$ *has a* 3*-SSG of order* $3^2 = 9$

and *number of* $3\text{-}SSG = 1 + 3t,\ t = 0,1,2,\ldots$ *and* $1 + 3t \mid 2^4.3^2$

$\Rightarrow$ *number of* $3\text{-}SSG = 1 + 3t,\ t = 0,1,2,\ldots$ *and* $1 + 3t \mid 2^4$, *Since* $\gcd(1+3t,\ 3^2) = 1$

$\Rightarrow$ *number of* $3\text{-}SSG = 1 + 3t,\ t = 0,1,2,\ldots$ *and* $1 + 3t \mid 16$

$\Rightarrow$ *number of* $3\text{-}SSG = 1,4,16.\quad t = 0,1,5.$

2.6 Theorem: A sylow p-subgroup of a finite group is unique if and only if it is normal. **Proof:** Let H is unique sylow p-subgroup of a finite group G such that O((H)= P^m, where P^m |*O(G)* and $P^{m+1} \nmid$ *O(G)* . Let $x \in G$ be any element of G then xHx^{-1} is a subgroup of G and $O(xHx^{-1})=O(H)= P^m \Rightarrow O(xHx^{-1})= P^m$, where P^m |*O(G)* and $P^{m+1} \nmid O(G)$. Thus xHx^{-1} is a sylow p-subgroup of G for all $x \in G$. Since H is the only sylow p-subgroup of G. Therefore $H= xHx^{-1}\ \forall\ x \in G$. Hence H is a normal subgroup of G.

Conversely let H is normal subgroup of G. Therefore

$$H= xHx^{-1} \ \forall\ x \in G \qquad \ldots(1)$$

Let K is any other sylow p-subgroup of G. Then by the second sylow theorem H and K are conjugate in G. i.e., $K= xHx^{-1}$ for some $x \in G$ …(2)

From equation (1) and (2) we have $H=K$. Hence H is a unique sylow p-subgroup of G.

APPLICATIONS OF SYLOW'S THEOREM

3.1 A group of order even perfect number is not a simple group.

Since order of group is an even perfect number .Therefore, let $O(G)=2^{p-1}\left(2^p-1\right)$, where p is a prime number. By First sylow theorem G has sylow (2^p-1)- subgroup and sylow $2^{(p-1)}$)- subgroup. *The number* n_{2^p-1} *of* $\left(2^p-1\right)-SSG$ *is given by*

$$n_{2^p-1} = 1+(2^p-1)k,\ k=0,1,2,...,\quad and\ n_{2^p-1}\,\middle|\,O(G)$$

$$\Rightarrow n_{2^p-1} = 1+(2^p-1)k,\ k=0,1,2,...,\quad and\ 1+(2^p-1)k\,\middle|\,2^{p-1}\left(2^p-1\right)$$

$$\Rightarrow n_{2^p-1} = 1+(2^p-1)k,\ k=0,1,2,...,\quad and\ 1+(2^p-1)k\,\middle|\,2^{p-1}$$

$$\Rightarrow n_{2^p-1} = 1,\ k=0.$$

$$\Rightarrow n_{2^p-1} = 1.$$

Thus, there exists exactly one 2^p-1 -SSG say H, where $O(H)=2^p-1$. Therefore H is a normal subgroup of order 2^p-1. Since group G has a proper normal subgroup. Hence G is not simple.

3.1.1 Examples: A group of order 28 is not simple.

Here $O(G)=28=2^2.7=2^{3-1}(2^3-1)$

By First sylow theorem G has sylow 2- *subgroup and sylow* 7-*subgroup.*

The number n_7 *of* $7-SSG$ *is given by*

$$n_7 = 1+7k,\ k=0,1,2,...,\quad and\ n_7\,|\,O(G)$$

$$\Rightarrow n_7 = 1+7k,\ k=0,1,2,...,\quad and\ 1+7k\,|\,2^2.7$$

$$\Rightarrow n_7 = 1+7k,\ k=0,1,2,...,\quad and\ 1+7k\,|\,2^2$$

$$\Rightarrow n_7 = 1,\ k=0.$$

$$\Rightarrow n_7 = 1.$$

Thus, there exists exactly one 7-*SSG, say H, where* $O(H)=7$. *Therefore H is a normal subgroup of order* 7. *Since group G has a proper normal subgroup of order* 7. *Hence G is not simple.*

3.1.2 Examples:

The groups of order $496 = 2^{5\text{-}1}\left(2^5 - 1\right) = 2^4.31$

and $8128 = 2^{7\text{-}1}\left(2^7 - 1\right) = 2^6.127$ *are not simple.*

3.2 A group G of order p.q (p and q are distinct primes) such that p < q and p does is divide (q-1) is not a simple group.

Here $O(G) = p.q$

By First sylow theorem G has sylow p *- subgroup and sylow* q *- subgroup.*

The number n_p *of* $p - SSG$ *is given by*

$$n_p = 1 + pk, \; k = 0,1,2,..., \quad \text{and } n_p \mid O(G)$$
$$\Rightarrow n_p = 1 + pk, \; k = 0,1,2,..., \quad \text{and } 1 + pk \mid p.q$$
$$\Rightarrow n_p = 1 + pk, \; k = 0,1,2,..., \quad \text{and } 1 + pk \mid q$$
$$\Rightarrow \text{either } n_p = 1 \quad \text{or} \quad n_p = q$$

if $n_p = q$

$$\Rightarrow 1 + pk = q$$
$$\Rightarrow pk = q - 1$$
$$\Rightarrow p \mid (q-1) \text{ , which is a contradiction.}$$

Therefore we have $n_p = 1.$

Thus, there exists exactly one p *- SSG, say* H*, where* $O(H) = p$*. Therefore* H *is a normal subgroup of order* p *.*

The number n_q *of* $q - SSG$ *is given by*

$$n_q = 1 + qk, \; k = 0,1,2,..., \quad \text{and } n_q \mid O(G)$$
$$\Rightarrow n_q = 1 + qk, \; k = 0,1,2,..., \quad \text{and } 1 + qk \mid p.q$$
$$\Rightarrow n_q = 1 + qk, \; k = 0,1,2,..., \quad \text{and } 1 + qk \mid p$$
$$\Rightarrow n_q = 1, \quad \text{since } q > p$$

Therefore we have $n_q = 1.$

Thus, there exists exactly one q *- SSG, say* K*, where* $O(K) = q$*. Therefore* K *is a normal subgroup of order* q*.*

Since G have two proper normal subgroups H and K of order p and q respectively. Hence G is not simple.

3.2.1 Examples: A group of order 33 is not simple.

Here $O(G) = 33 = 3.11$ *where* 3, 5 *are distinct primes and* 3 *does not* $(11\text{-}1) = 10.$

Therefore G is not simple.

3.3 A group G of order 2p (p being a prime) is not a simple group.

Here $O(G) = 2p$

By First sylow theorem G has sylow p - subgroup . The number n_p *of* $p - SSG$ *is given by*

$$n_p = 1 + pk, \ k = 0,1,2,..., \quad and \ n_p \mid O(G)$$
$$\Rightarrow n_p = 1 + pk, \ k = 0,1,2,..., \quad and \ 1 + pk \mid 2p$$
$$\Rightarrow n_p = 1 + pk, \ k = 0,1,2,..., \quad and \ 1 + pk \mid 2$$
$$\Rightarrow n_p = 1$$

Therefore we have $n_p = 1.$

Thus, there exists exactly one p - SSG, say H, where $O(H) = p$. *Therefore H is a normal subgroup of order p . Since group G has a proper normal subgroup H of order p. Hence G is not simple.*

3.4 A group G of order pqr and p < q < r (p, q and r are primes) is not a simple group.

Here $O(G) = pqr$

By First sylow theorem G has sylow p - subgroup, sylow q - subgroup and sylow p - subgroup .

The number n_p *of* $p - SSG$ *is given by*

$$n_p = 1 + pk, \ k = 0,1,2,..., \quad and \ n_p \mid O(G)$$
$$\Rightarrow n_p = 1 + pk, \ k = 0,1,2,..., \quad and \ 1 + pk \mid pqr$$
$$\Rightarrow n_p = 1 + pk, \ k = 0,1,2,..., \quad and \ 1 + pk \mid qr$$
$$\Rightarrow n_p = 1, \ q, \ r, \ qr$$

The number n_q *of* $q - SSG$ *is given by*

$$n_q = 1 + qk, \ k = 0,1,2,..., \quad and \ n_q \mid O(G)$$
$$\Rightarrow n_q = 1 + qk, \ k = 0,1,2,..., \quad and \ 1 + qk \mid pqr$$
$$\Rightarrow n_q = 1 + qk, \ k = 0,1,2,..., \quad and \ 1 + qk \mid pr$$
$$\Rightarrow n_q = 1, \ r, \ pr, \ \not{p}$$
$$\Rightarrow n_q = 1, \ r, \ pr \quad \text{since } p < q.$$

The number n_r *of* $r - SSG$ *is given by*

$$n_r = 1 + rk, \ k = 0,1,2,..., \quad and \ n_r \mid O(G)$$
$$\Rightarrow n_r = 1 + rk, \ k = 0,1,2,..., \quad and \ 1 + rk \mid pqr$$
$$\Rightarrow n_r = 1 + rk, \ k = 0,1,2,..., \quad and \ 1 + rk \mid pq$$
$$\Rightarrow n_r = 1, \ \not{p}, \ \not{q}, \ pq$$
$$\Rightarrow n_r = 1, \ pq \quad \text{since } p < q < r .$$

Let non these SSG is normal then

$$n_p = q, \ r, \ qr$$
$$n_q = r, \ pr$$
$$and \quad n_r = pq$$

The minimum number of SSG is given by

$$n_p = q$$
$$n_q = r$$
$$and \quad n_r = pq$$

Now, the minimum number elements in G

$$= q(p-1) + r(q-1) + pq(r-1)$$

Thus $\quad q(p-1) + r(q-1) + pq(r-1) + 1 \le pqr$

$$\Rightarrow pq - q + qr - r + pqr - pq + 1 \le pqr$$
$$\Rightarrow -q + qr - r + 1 \le 0$$

$\Rightarrow r(q-1)-1(q-1)\leq 0$

$\Rightarrow (q-1)(r-1)\leq 0$ *which is a contradiction. Since r and q are primes.*

For example $(2-1)(3-1)\nleq 0.$

That our initial assumption that non any SSG is not normal is wrong. Therefore some SSG is normal. Hence G is not simple.

3.5 Some Other Examples:

Order of Group	Prime Factor	Special Feature
8	2^3	Prime Cubed
12	$2^2 \cdot 3$	A_4
18	$2\cdot 3^2$	Normal Sylow 3
20	$2^2 \cdot 5$	Normal Sylow 5
27	3^3	Prime cubed
28	$2^2\cdot 7$	Normal Sylow 7
30	$2\cdot 3\cdot 5$	Normal Sylow 3 or 5
42	$2\cdot 3\cdot 7$	$Z_2 \to U(7)\times U(3)$
44	$2^2\cdot 11$	Normal Sylow 11
50	$2\cdot 5^2$	Normal Sylow 5
52	$2^2\cdot 13$	Normal Sylow 13
63	$3^2\cdot 7$	Normal Sylow 7
68	$2^2\cdot 17$	Normal Sylow 17
70	$2\cdot 5\cdot 7$	Normal Sylow 5 or 7
75	$3\cdot 5^2$	Normal Sylow 5
76	$2^2\cdot 19$	Normal Sylow 19
78	$2\cdot 3\cdot 13$	$Z_2 \to U(13)\times U(13)$
92	$2^2\cdot 23$	Normal Sylow 23
98	$2\cdot 7^2$	Normal Sylow 7
99	$3^2\cdot 11$	Normal Sylow 3 or 11
117	$3^2\cdot 13$	Normal Sylow 3 or 13
153	$3^2\cdot 17$	Normal Sylow 3 or 17
171	$3^2\cdot 19$	Normal Sylow 3 or 19
207	$3^2\cdot 23$	Normal Sylow 3 or 23
261	$3^2\cdot 29$	Normal Sylow 3 or 29
279	$3^2\cdot 31$	Normal Sylow 3 or 31
333	$3^2\cdot 37$	Normal Sylow 3 or 37

REFERENCES

1. I. N. Herstein. Topics in Algebra. John Wiley & Sons. 1975
2. Jack Schmidt."Sylow structure of finite groups" International Mathematical Forum Vol.09291-295, sep. 2-2002.
3. D. Dimmit and R. Foote. Abstract Algebra. John Wiley and Sons, 2004.

4. W. C. Waterhouse, The early proofs of Sylow's theorem, Arch. Hist. Exact Sci. 21 (1979/80), 279–290.
5. Paul.E.Bland: The Basics of Abstract Algebra. W.H.Freeman and Company, United States of America, 2001.
6. I.N. Herstein: Abstract Algebra, 3rd ed. Prentice Hall, Upper Saddle River, NJ, 1995.
7. Joseph.J. Rotman: A First Course in Absract Algebra, 2nd ed. Prentice Hall, Upper Saddle River, NJ, 2000.
8. Joseph.J.Rotman: Advanced Modern Algebra. Prentice Hall, Upper Saddle River, NJ, 2002.
9. Joseph.J.Rotman: An Introduction to the Theory of Groups, Springer-Verlag, New York, 1995.
10. John S. Rose: A course on Group Theory. Dover Publications, Cambridge, England, 2012. 51
11. "Contemporary Abstract Algebra" by Joseph A. Gallian.
12. "A Course in Abstract Algebra" by V.K. Khanna& S.K.